Solutions Assignment 2 for Grading
Department of Economics
Carleton University
ECON 2400C

Instructor: Marcel Voia

Solutions Exercises:

Problems from the book:

1. Exercise 12.2 page 355, Problem 1 (points: a and d) **(10 points each)** including the SOC results. In the solutions, problem 2 becomes the SOC for problem 1, therefore after problem "1" we go directly to problem 3:

Answer: a.1) The Lagrangian is:

$$L(x, y, \lambda) = xy - \lambda(x + 2y - 2)$$

a.2) FOC

$$\frac{\partial L}{\partial x} = y - \lambda = 0$$

$$\frac{\partial L}{\partial y} = x - 2\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = x + 2y - 2 = 0$$

From equation 1 and 2 we have that

$$\lambda = y = \frac{x}{2}$$

Replacing $y = \frac{x}{2}$ in 3 we get

$$x + 2\frac{x}{2} = 2 \Leftrightarrow x^* = 1, y^* = 1/2, \lambda^* = 1/2.$$

Use the bordered Hessian to determine whether the stationary value obtained at point 1 is a maximum or a minimum

a.3) SOC The Border Hessian is defined as:

$$BH = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

Because the determinant of the bordered hessian is positive, the solution solve the maximization problem.

d.1) The Lagrangian is:

$$L(x, y, \lambda) = 7 - y + x^2 - \lambda(x + y)$$

d.2) FOC

$$\frac{\partial L}{\partial x} = 2x - \lambda = 0$$
$$\frac{\partial L}{\partial y} = -1 - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = x + y = 0$$

From equation 1 we have that $x = \lambda = 2$
From equation 2, we have that $\lambda = -1$

$$\Rightarrow x^* = -1/2$$

From equation 3, we have

$$y^* = 1/2$$

The students may say that because $\lambda < 0$ there is no optimal solution. YOu do not need to penalize them

Use the bordered Hessian to determine whether the stationary value obtained at point 1 is a maximum or a minimum
d.3) SOC The Border Hessian is defined as:

$$BH = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Because the determinant of the bordered hessian is negative, the solution solve the minimization problem.

3. **(10 points)** Exercise 12.5 page 382, Problem 1: use the constrained from page 374, ie use as a constraint the following: xPx +yPy=B

$$U(x, y) = (x + 2)(y + 1)$$

subject to

$$xP_x + yP_y = B$$

3.1) The Lagrangian is:

$$L(x, y, \lambda) = (x + 2)(y + 1) - \lambda(xP_x + yP_y - B)$$

3.2) FOC

$$\frac{\partial L}{\partial x} = y + 1 - \lambda P_x = 0$$
$$\frac{\partial L}{\partial y} = x + 2 - \lambda P_y = 0$$

$$\frac{\partial L}{\partial \lambda} = xP_x + yP_y - B = 0$$

Taking the ratio of the first two equation we get

$$yP_y = xP_x + 2P_x - P_y$$

$$y = \frac{P_x(x+2)}{P_y} - 1$$

$$xP_x + xP_x + 2P_x - P_y = B$$

$$x^* = \frac{B + P_y - 2P_x}{2P_x}$$

$$y^* = (x+2)\frac{P_x}{P_y} - 1 = (\frac{B + P_y - 2P_x}{2P_x} + 2)\frac{P_x}{P_y} - 1$$

$$y^* = \frac{B - P_y + 2P_x}{2P_y}$$

3.3) SOC The Border Hessian is defined as:

$$BH = \begin{pmatrix} 0 & P_x & P_y \\ P_x & 0 & 1 \\ P_y & 1 & 0 \end{pmatrix}$$

$$detBH = 2P_xP_y > 0$$

Because the determinant of the bordered hessian is positive, the solution solve the maximization problem.

4. **(10 points all. 3 points each. 1 point to start)** Exercise 12.6 page 389, Problem 1: (points a,c and e)

a) To check the homogeneity of $f(.)$, multiply the inputs with a constant t and check how the output looks like. We have

$$f(tx, ty) = \sqrt{txty} = \sqrt{t^2xy} = t\sqrt{xy} = tf(xy)$$

The function is homogeneous of degree 1.

b) To check the homogeneity of $f(.)$, multiply the inputs with a constant t and check how the output looks like. We have

$$f(tx, ty) = t^3x^3 - t^2xy + t^3y^2$$

The function is not homogeneous.

c) To check the homogeneity of $f(.)$, multiply the inputs with a constant t and check how the output looks like. We have

$$f(tx, ty, tw) = \frac{t^3xy^2}{tw} + 2t^2xw = t^2(\frac{xy^2}{w} + 2xw) = t^2f(x, y, w)$$

The function is homogeneous of degree 2.

5. **(10 points)** Exercise 12.7 page 401, for Problem 7 compute the elasticity of substitution between K and L (K/L)

$$MRTS_{LK} = \frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = \frac{1-\delta}{\delta}\left(\frac{K}{L}\right)^{p+1}$$

From here we can solve for $\frac{K}{L}$

$$\frac{K}{L} = \left(MRTS_{LK}\frac{\delta}{1-\delta}\right)^{\frac{1}{p+1}}$$

The elasticity of substitution is found as

$$\sigma_{MRTS}\left(\frac{K}{L}\right) = \frac{\partial\left(\frac{K}{L}\right)}{\partial MRTS_{LK}}\frac{MRTS_{LK}}{\left(\frac{K}{L}\right)}$$

$$\sigma_{MRTS}\left(\frac{K}{L}\right) = \frac{1}{p+1}\frac{MRTS_{LK}^{-p/(p+1)}MRTS_{LK}}{MRTS_{LK}^{1/(p+1)}} = \frac{1}{p+1}.$$

6. **(15 points)**

$$min_{x,y,z}(x+4y+3z)\ st\ x^2+2y^2+\frac{1}{3}z^2 = b$$

with $b > 0$ Check the Second Order Conditions for your solution(s).

Solution:

Define the Lagrangian

$$L(x,y,z) = x+4y+3z-\lambda(x^2+2y^2+\frac{1}{3}z^2-b)$$

The solution of the problem is found by solving

$$L'_x(x,y,z) = 1-2\lambda x = 0$$

$$L'_y(x,y,z) = 4-4\lambda y = 4(1-\lambda y) = 0$$

$$L'_z(x,y,z) = 3-\frac{2}{3}\lambda z = 0$$

$$L'_\lambda(x,y) = x^2+2y^2+\frac{1}{3}z^2-b = 0$$

From the first two equations we have

$$1-2\lambda x = 1-\lambda y \Rightarrow y^* = 2x^*$$

From the second equation we have

$$\lambda = \frac{1}{y}$$

From equation 3 we have

$$3 - \frac{2}{3}\lambda z = 0 \Leftrightarrow 3 - \frac{2}{3}(\frac{1}{y})z = 0 \Leftrightarrow \frac{2}{3}(\frac{1}{y})z = 3 \Rightarrow z^* = \frac{9y^*}{2} = 9x^*.$$

Replacing $y^* = 2x^*$ and $z^* = 9x^*$ in the fourth equation, we solve for x^*

$$x^2 + 8x^2 + \frac{1}{3}81x^2 - b = 0 \Leftrightarrow x^2(1 + 8 + 27) = b \Leftrightarrow x^2 = \frac{b}{36}$$

which gives the following solutions for $x^* = +/-\frac{1}{6}\sqrt{b}$. If $x^* = -\frac{1}{6}\sqrt{b}$, equation 1 of the FOC gives a negative λ, which is impossible, therefore the only viable solution for x^* is $x^* = \frac{1}{6}\sqrt{b}$. Given x^*, the solution of the problem is

$x^* = \frac{1}{6}\sqrt{b}$, $y^* = \frac{1}{3}\sqrt{b}$ and $z^* = \frac{3}{2}\sqrt{b}$.

$$\frac{\partial^2 L}{\partial x^2} = -2\lambda$$

$$\frac{\partial^2 L}{\partial x^2} = -4\lambda$$

$$\frac{\partial^2 L}{\partial x^2} = -\frac{2}{3}\lambda$$

$$\frac{\partial^2 L}{\partial xy} = \frac{\partial^2 L}{\partial yx} = \frac{\partial^2 L}{\partial xz} = \frac{\partial^2 L}{\partial zy} = 0$$

$$BorderedHessian = \begin{pmatrix} 0 & 2x & 4y & \frac{2}{3}z \\ 2x & -2\lambda & 0 & 0 \\ 4y & 0 & -4\lambda & 0 \\ \frac{2}{3}z & 0 & 0 & -\frac{2}{3}\lambda \end{pmatrix}$$

The determinant of the Bordered Hessian is negative, which means the solution is a min.

7. **(15 points)**

$$max_{x,y}(x^2 + 2y^2 - x) \ st \ x^2 + y^2 \leq 1$$

a) $L(x, y, \lambda) = x^2 + 2y^2 - x - \lambda(x^2 + y^2 - 1)$

b)FOC

$$\frac{\partial L}{\partial x} = 2x - 1 - 2x\lambda = 0$$

$$\frac{\partial L}{\partial y} = 4y - 2\lambda y = 0$$

Discussion:

- If $\lambda = 0$ we have the constraint $x^2 + y^2 - 1 < 0$ When $\lambda = 0 \Rightarrow y_1^* = 0$ from the second equation of the FOC and $x_1^* = 1/2$ from the first equation of FOC.
  Now check if (x_1^*, y_1^*) satisfy the inequality constraint, we have

$$\frac{1}{4} - 1 < 0$$

  True, which means that the pair $(x_1^* = 1/2, y_1^* = 0)$ is a potential solution

- if $\lambda > 0$ use the constraint with equality

$$x^2 + y^2 - 1 = 0$$

  From the second equation of the FOC we have that $\lambda = 2 > 0$.

  From the first equation of FOC, we have $2x - 1 - 4x = 0 \Rightarrow x_2^* = -\frac{1}{2}$

  From the equality costraint

$$\frac{1}{4} + y^2 - 1 = 0$$

  $\Rightarrow y^2 = \frac{3}{4} \Leftrightarrow y_{2,3}^* = \pm\frac{\sqrt{3}}{2}$ Therefore when $\lambda = 2$ we haveanother 2 potential solutions:

  $(x_2^* = -1/2, y_2^* = \frac{\sqrt{3}}{2})$ and $(x_3^* = -1/2, y_3^* = -\frac{\sqrt{3}}{2})$

Check the Second Order Conditions for your solution(s)

$$\frac{\partial^2 L}{\partial x^2} = 2 - 2\lambda$$

$$\frac{\partial^2 L}{\partial y^2} = 4 - 2\lambda$$

$$\frac{\partial^2 L}{\partial x \partial y} = \frac{\partial^2 L}{\partial y \partial x} = 0$$

$$Hessian = \begin{pmatrix} 2 - 2\lambda & 0 \\ 0 & 4 - 2\lambda \end{pmatrix}$$

Discussion:
If $\lambda = 0$, the Lagrangian is convex.

$$\frac{\partial^2 L}{\partial x^2} > 0,$$

$$\frac{\partial^2 L}{\partial y^2} > 0$$

and the determinant of the hessian is positive.

If $\lambda > 0$, $\lambda = 2$

$$\frac{\partial^2 L}{\partial x^2} = 2 - 2\lambda = -2$$

$$\frac{\partial^2 L}{\partial y^2} = 4 - 2\lambda = 0$$

$$Hessian = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow$ the determinant of the Hessian is 0. $\Rightarrow$ that $(x_2^* = -1/2, y_2^* = \frac{\sqrt{3}}{2})$ and $(x_3^* = -1/2, y_3^* = -\frac{\sqrt{3}}{2})$ can be both local max.