MAT 1332, Winter 2014, Assignment 6
Due Friday March 7 by 3:00pm.
Late assignments will not be accepted; nor will unstapled assignments.
Professors in the math department will not lend you a stapler; do not ask for one.

Instructor (circle one): Robert Smith? Frithjof Lutscher Catalin Rada

1. Consider the complex numbers

$$z = 2 + 5\mathrm{i} \qquad \text{and} \qquad w = -3 + 4\mathrm{i}.$$

Calculate the following

(a) $z + \bar{w}$

Solution: $= (2 + 5i) + (-3 - 4i) = -1 + i$,

(b) $|z + w|$

Solution: $= |(2 + 5i) + (-3 + 4i)| = |-1 + 9i| = \sqrt{(-1)^2 + (9)^2} = \sqrt{82}$,

(c) zw

Solution: $= (2 + 5i)(-3 + 4i) = -6 - 15i + 8i + 20i^2 = -26 - 7i$,

(d) w/z

Solution: $= \frac{-3+4i}{2+5i} = \frac{(-3+4i)(2-5i)}{(2+5i)(2-5i)} = \frac{-6+8i+15i-20i^2}{4+25} = \frac{14+23i}{29} = \frac{14}{29} + \frac{23}{29}i$

2. A manufacturer sells three types of fertilizers, which are mixtures of three products A, B and C. The fertilizer of the type I contains, by unit, 10 kg of A, 30 kg of B and 60 kg of C; that of type II, 20 kg of A, 30 kg of B and 50 kg of C; that of type III, 50 kg of A and 50 kg of C. There are currently in stock 2000 kg of A, 900 kg of B and 3100 kg of C. How many units of each type of fertilizer does the manufacturer have to produce if they want to dispose of all the raw materials in stock?

Solution: Let x_1 be the number of type I fertilizer, x_2 be the number of type II fertilizer and x_3 be the number of type III fertilizer. So, after the above indications and inventory of the company, the amount of each product A, B and C necessary for the total production is given by the system of linear equations:

$$\begin{aligned} \text{product A}: \quad 10x_1 + 20x_2 + 50x_3 &= 2000 \\ \text{product B}: \quad 30x_1 + 30x_2 + 0x_3 &= 900 \\ \text{product C}: \quad 60x_1 + 50x_2 + 50x_3 &= 3100 \end{aligned}$$

We use row reduction to solve the system:

$$\begin{bmatrix} 10 & 20 & 50 & | & 2000 \\ 30 & 30 & 0 & | & 900 \\ 60 & 50 & 50 & | & 3100 \end{bmatrix} \underset{\substack{\left(\frac{1}{10}\right) R_1 \to R_1 \\ \left(\frac{1}{30}\right) R_2 \to R_2 \\ \left(\frac{1}{10}\right) R_3 \to R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 5 & | & 200 \\ 1 & 1 & 0 & | & 30 \\ 6 & 5 & 5 & | & 310 \end{bmatrix}$$

$$\underset{\substack{R_2 - R_1 \to R_2 \\ R_3 - (6)R_1 \to R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 5 & | & 200 \\ 0 & -1 & -5 & | & -170 \\ 0 & -7 & -25 & | & -890 \end{bmatrix}$$

$$\underset{R_3 - (7)R_2 \to R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 5 & | & 200 \\ 0 & -1 & -5 & | & -170 \\ 0 & 0 & 10 & | & 300 \end{bmatrix}$$

$$\underset{\substack{(-1)R_2 \to R_2 \\ \left(\frac{1}{10}\right) R_3 \to R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 5 & | & 200 \\ 0 & 1 & 5 & | & 170 \\ 0 & 0 & 1 & | & 30 \end{bmatrix}$$

$$\underset{\substack{R_1 - (5)R_3 \to R_1 \\ R_2 - (5)R_3 \to R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & | & 50 \\ 0 & 1 & 0 & | & 20 \\ 0 & 0 & 1 & | & 30 \end{bmatrix}$$

$$\underset{R_1 - (2)R_2 \to R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & | & 10 \\ 0 & 1 & 0 & | & 20 \\ 0 & 0 & 1 & | & 30 \end{bmatrix}$$

So we have to produce 10 units of fertilizer type I, 20 units of fertilizer type II and 30 units of fertilizer type III to exhaust the inventory of the company.

3. For each of the following matrices, find its eigenvalues and the corresponding eigenvectors.

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 9 \\ -1/2 & -1 \end{bmatrix}$$

Solution:

(a) For $A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$.

- The characteristic polynomial is A, $p(\lambda)$:

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det\left(\begin{bmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix}\right) \\ &= (5-\lambda)(2-\lambda) - (2)(2) \\ &= 10 - 7\lambda + \lambda^2 - 4 \\ &= \lambda^2 - 7\lambda + 6 \\ &= (\lambda - 6)(\lambda - 1) \end{aligned}$$

- The eigenvalues satisfy $p(\lambda)$:

$$\begin{aligned} p(\lambda) &= 0 \\ (\lambda - 6)(\lambda - 1) &= 0 \\ \lambda &= 6 \text{ or } 1 \end{aligned}$$

- Thus the eigenvalues of A are $\lambda = 6$ and $\lambda = 1$.

- To find the eigenvectors of A associated with the eigenvalue $\lambda = 6$ we reduce the associated augmented matrix system $[A - (6)I|0]$:

$$\begin{aligned} \begin{bmatrix} A - 6I & | & 0 \end{bmatrix} &= \begin{bmatrix} -1 & 2 & | & 0 \\ 2 & -4 & | & 0 \end{bmatrix} \\ &\overset{R_2+2R_1 \to R_2}{\rightsquigarrow} \begin{bmatrix} -1 & 2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \\ &\overset{(-1)R_1 \to R_1}{\rightsquigarrow} \begin{bmatrix} 1 & -2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \end{aligned}$$

Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \end{bmatrix} ; s \neq 0$$

are the eigenvectors associated with the eigenvalue $\lambda = 6$.

- To find the eigenvectors of A associated with the eigenvalue $\lambda = 1$ we reduce the associated augmented matrix system $[A - (1)I|0]$:

$$\begin{aligned} \begin{bmatrix} A - I & | & 0 \end{bmatrix} &= \begin{bmatrix} 4 & 2 & | & 0 \\ 2 & 1 & | & 0 \end{bmatrix} \\ &\overset{(\frac{1}{4})R_1 \to R_1}{\rightsquigarrow} \begin{bmatrix} 1 & \frac{1}{2} & | & 0 \\ 2 & 1 & | & 0 \end{bmatrix} \\ &\overset{R_2-(2)R_1 \to R_2}{\rightsquigarrow} \begin{bmatrix} 1 & \frac{1}{2} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \end{aligned}$$

Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} ; s \neq 0$$

are the eigenvectors associated with the eigenvalue $\lambda = 1$.

(b) For $B = \begin{bmatrix} 2 & 9 \\ -1/2 & -1 \end{bmatrix}$.

- We find the characteristic polynomial of B, $p(\lambda)$:

$$\begin{aligned} p(\lambda) &= \det(B - \lambda I) \\ &= \det\left(\begin{bmatrix} 2-\lambda & 9 \\ -\frac{1}{2} & -1-\lambda \end{bmatrix}\right) \\ &= (2-\lambda)(-1-\lambda) - (9)\left(-\frac{1}{2}\right) \\ &= -2 - \lambda + \lambda^2 + \frac{9}{2} \\ &= \lambda^2 - \lambda + \frac{5}{2} \end{aligned}$$

- We use the quadratic formula to find $p(\lambda)$:

$$\begin{aligned} \lambda &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)\left(\frac{5}{2}\right)}}{2(1)} \\ &= \frac{1 \pm \sqrt{-9}}{2} \\ &= \frac{1}{2} \pm \frac{3}{2}i \end{aligned}$$

- Thus the eigenvalues of B are $\lambda = \frac{1}{2} + \frac{3}{2}i$ and $\lambda = \frac{1}{2} - \frac{3}{2}i$.

- To find the eigenvectors of B associated with the eigenvalue $\lambda = \frac{1}{2} + \frac{3}{2}i$ we solve the associated augmented system $[B - \left(\frac{1}{2} + \frac{3}{2}\right) I | 0]$:

$$\begin{aligned}
\left[B - \left(\tfrac{1}{2} - \tfrac{3}{2}\right) I \quad | \quad 0\right] &= \begin{bmatrix} \frac{3}{2} - \frac{3}{2}i & 9 & | & 0 \\ -\frac{1}{2} & -\frac{3}{2} - \frac{3}{2}i & | & 0 \end{bmatrix} \\
&\overset{R_1 \leftrightarrow R_2}{\rightsquigarrow} \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} - \frac{3}{2}i & | & 0 \\ \frac{3}{2} - \frac{3}{2}i & 9 & | & 0 \end{bmatrix} \\
&\overset{(-2)R_1 \to R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 3 + 3i & | & 0 \\ \frac{3}{2} - \frac{3}{2}i & 9 & | & 0 \end{bmatrix} \\
&\overset{R_2 - \left(\frac{3}{2} - \frac{3}{2}i\right)R_1 \to R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 3 + 3i & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} -3 - 3i \\ 1 \end{bmatrix}; s \neq 0$$

are the eigenvectors associated with $\lambda = \frac{1}{2} + \frac{3}{2}i$.

- To find the eigenvectors of B associated with the eigenvalue $\lambda = \frac{1}{2} - \frac{3}{2}i$ we use the proprty of complex conjugates (or solve). Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} -3 + 3i \\ 1 \end{bmatrix}; s \neq 0$$

are the eigenvectors associated with $\lambda = \frac{1}{2} - \frac{3}{2}i$.

4. For each of the following matrices, find its eigenvalues and the corresponding eigenvecotrs

$$A = \begin{bmatrix} 5 & 0 & 3 \\ 0 & 2 & 0 \\ 7 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$$

Solution:

(a) For $A = \begin{bmatrix} 5 & 0 & 3 \\ 0 & 2 & 0 \\ 7 & 0 & 1 \end{bmatrix}$.

- We find the characteristic polynomial of A, $p(\lambda)$:

$$\begin{aligned}
p(\lambda) &= \det(A - \lambda I) \\
&= \det\left(\begin{bmatrix} 5-\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ 7 & 0 & 1-\lambda \end{bmatrix}\right) \\
&= (5-\lambda)(2-\lambda)(1-\lambda) - 3(2-\lambda)(7) \\
&= (2-\lambda)((5-\lambda)(1-\lambda) - 21) \\
&= (2-\lambda)(5 - 6\lambda + \lambda^2 - 21) \\
&= (2-\lambda)(\lambda^2 - 6\lambda - 16) \\
&= (2-\lambda)(\lambda - 8)(\lambda + 2)
\end{aligned}$$

- The eigenvalues are the roots of $p(\lambda)$:

$$\begin{aligned}
p(\lambda) &= 0 \\
(2-\lambda)(\lambda - 8)(\lambda + 2) &= 0 \\
\lambda &= -2, 2 \text{ or } 8
\end{aligned}$$

- Thus the eigenvalues of A are $\lambda = -2$, $\lambda = 2$ and $\lambda = 8$.

- To find the eigenvectors of A associated with the eigenvalue $\lambda = -2$ we solve the augmented system $[A - (-2)I|0]$:

$$\begin{bmatrix} A+2I & | & 0 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 3 & | & 0 \\ 0 & 4 & 0 & | & 0 \\ 7 & 0 & 3 & | & 0 \end{bmatrix} \overset{R_3 - R_1 \to R_3}{\rightsquigarrow} \begin{bmatrix} 7 & 0 & 3 & | & 0 \\ 0 & 4 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \overset{\substack{\left(\frac{1}{7}\right) R_1 \to R_1 \\ \left(\frac{1}{4}\right) R_2 \to R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & \frac{3}{7} & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} -\frac{3}{7} \\ 0 \\ 1 \end{bmatrix} ; s \neq 0$$

are the eigenvectors associated with $\lambda = -2$.

- To find the eigenvectors of A associated with the eigenvalue $\lambda = 2$ we solve the augmented system $[A - (2)I|0]$:

$$\begin{bmatrix} A-2I & | & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 7 & 0 & -1 & | & 0 \end{bmatrix} \overset{\left(\frac{1}{3}\right) R_1 \to R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 7 & 0 & -1 & | & 0 \end{bmatrix} \overset{R_3 - (7) R_1 \to R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & -8 & | & 0 \end{bmatrix} \overset{R_2 \leftrightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 0 & -8 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \overset{\left(-\frac{1}{8}\right) R_2 \to R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \overset{R_1 - R_2 \to R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} ; s \neq 0$$

are the eigenvectors associated with $\lambda = -2$.

- To find the eigenvectors of A associated with the eigenvalue $\lambda = 8$ we solve the augmented system $[A-(8)I|0]$:

$$\begin{aligned}
\left[A - 8I \quad | \quad 0\right] &= \begin{bmatrix} -3 & 0 & 3 & | & 0 \\ 0 & -6 & 0 & | & 0 \\ 7 & 0 & -7 & | & 0 \end{bmatrix} \\
\underset{\rightsquigarrow}{\substack{\left(-\frac{1}{3}\right) R_1 \to R_1 \\ \left(-\frac{1}{6}\right) R_2 \to R_2}} &\quad \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 7 & 0 & -7 & | & 0 \end{bmatrix} \\
\underset{\rightsquigarrow}{R_3-(7)R_1 \to R_3} &\quad \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}
\end{aligned}$$

Thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} ; s \neq 0$$

are the eigenvectors associated with $\lambda = 8$.

(b) For $B = \begin{bmatrix} -1 & 2 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$.

- We find the characteristic polynomial of B, $p(\lambda)$:

$$\begin{aligned}
p(\lambda) &= \det(B - \lambda I) \\
&= \det\left(\begin{bmatrix} -1-\lambda & 2 & 0 \\ 4 & 1-\lambda & 0 \\ 0 & 0 & \sqrt{5}-\lambda \end{bmatrix}\right) \\
&= (-1-\lambda)(1-\lambda)(\sqrt{5}-\lambda) - (\sqrt{5}-\lambda)(4)(2) \\
&= (\sqrt{5}-\lambda)\left((-1-\lambda)(1-\lambda) - 8\right) \\
&= (\sqrt{5}-\lambda)(-1+\lambda^2-8) \\
&= (\sqrt{5}-\lambda)(\lambda^2-9) \\
&= (\sqrt{5}-\lambda)(\lambda-3)(\lambda+3)
\end{aligned}$$

- We find the roots of $p(\lambda)$:

$$\begin{aligned}
p(\lambda) &= 0 \\
(\sqrt{5}-\lambda)(\lambda-3)(\lambda+3) &= 0 \\
\lambda &= \sqrt{5}, 3 \text{ ou } -3
\end{aligned}$$

- The eigenvalues of B are $\lambda = \sqrt{5}$, $\lambda = 3$ and $\lambda = -3$.

- To find the eigenvectors of B associated with the eigenvalue $\lambda = \sqrt{5}$ we solve the augmented system $[B - (\sqrt{5})I|0]$:

$$\begin{bmatrix} B-(\sqrt{5})I & | & 0 \end{bmatrix} = \begin{bmatrix} -1-\sqrt{5} & 2 & 0 & | & 0 \\ 4 & 1-\sqrt{5} & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\overset{R_1 \leftrightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 4 & 1-\sqrt{5} & 0 & | & 0 \\ -1-\sqrt{5} & 2 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\overset{\left(\frac{1}{4}\right)R_1 \to R_1}{\rightsquigarrow} \begin{bmatrix} 1 & \frac{1-\sqrt{5}}{4} & 0 & | & 0 \\ -1-\sqrt{5} & 2 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\overset{R_2 + (1+\sqrt{5})R_1 \to R_2}{\rightsquigarrow} \begin{bmatrix} 1 & \frac{1-\sqrt{5}}{4} & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\overset{R_1 - \left(\frac{1-\sqrt{5}}{4}\right)R_2 \to R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ; s \neq 0$$

are the eigenvectors associated with $\lambda = \sqrt{5}$.

- To find the eigenvectors of B associated with the eigenvalue $\lambda = 3$ we solve the augmented system $[B - (3)I|0]$:

$$\begin{bmatrix} B-(3)I & | & 0 \end{bmatrix} = \begin{bmatrix} -4 & 2 & 0 & | & 0 \\ 4 & -2 & 0 & | & 0 \\ 0 & 0 & \sqrt{5}-3 & | & 0 \end{bmatrix}$$

$$\overset{R_2 + R_1 \to R_2}{\rightsquigarrow} \begin{bmatrix} -4 & 2 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & \sqrt{5}-3 & | & 0 \end{bmatrix}$$

$$\overset{R_2 \leftrightarrow R_3}{\rightsquigarrow} \begin{bmatrix} -4 & 2 & 0 & | & 0 \\ 0 & 0 & \sqrt{5}-3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\overset{\substack{\left(-\frac{1}{4}\right)R_1 \to R_1 \\ \left(\frac{1}{\sqrt{5}-3}\right)R_2 \to R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & -\frac{1}{2} & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} ; s \neq 0$$

are the eigenvectors associated with $\lambda = 3$.

- To find the eigenvectors of B associated with the eigenvalue $\lambda = -3$ we solve the augmented system $[B-(-3)I|0]$:

$$\begin{bmatrix} B+(3)I & | & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 & | & 0 \\ 4 & 4 & 0 & | & 0 \\ 0 & 0 & \sqrt{5}+3 & | & 0 \end{bmatrix}$$

$$\overset{R_2-(2)R_1\to R_2}{\rightsquigarrow} \begin{bmatrix} 2 & 2 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & \sqrt{5}+3 & | & 0 \end{bmatrix}$$

$$\overset{R_2\leftrightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 2 & 2 & 0 & | & 0 \\ 0 & 0 & \sqrt{5}+3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\overset{\substack{\left(\frac{1}{2}\right)R_1\to R_1 \\ \left(\frac{1}{\sqrt{5}+3}\right)R_2\to R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} ; s \neq 0$$

are the eigenvectors associated with $\lambda = -3$.

5. Perennial plants flower some years but not others. Suppose that a plant that is flowering one year will also flower in the following year with a probability of 80%. A plant that is not flowing in one year will flower the following year with a probability of 60%. Suppose that the total number of plants is kept constant over the years. Denote by x_n the number of plants on a certain meadow that flower in year n and by y_n the number of plants that do not flower.

(a) Write the transition matrix A so that the iteration

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = A \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

describes the number of flowering and non-flowering plants from one year to the next.

(b) Calculate the number of flowering plants in year 3 if in year zero there are 200 flowering plants and 100 non-flowering plants.

(c) Calculate the percentage of plants that are expected to flower in a given year in the long run.

Solution: The number of plants flowering in year $n+1$ equals the number of plants flowering in year n times the probability that they will flower again, plus the number of plants not flowering in year n times the probability that they will flower. In equations:

$$x_{n+1} = 0.8x_n + 0.6y_n.$$

Accordingly, the number of plants not flowering is

$$y_{n+1} = (1-0.8)x_n + (1-0.6)y_n.$$

In matrix notation

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}.$$

With initial conditions $x_0 = 200, y_0 = 100$ we find

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 200 \\ 100 \end{bmatrix} = \begin{bmatrix} 220 \\ 80 \end{bmatrix}.$$

Then

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 224 \\ 76 \end{bmatrix} \qquad \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 224.8 \\ 75.2 \end{bmatrix}.$$

Obviously, we cannot have fractions of plants flowering. Instead, we consider the percentage of plants flowering. In the long run, we have x_n, y_n approach the eigenvector of A with eigenvalue 1 (see result from class). This eigenvector satisfies

$$(A - I)v = 0, \qquad \begin{bmatrix} -0.2 & 0.6 \\ 0.2 & -0.6 \end{bmatrix} v = 0, \qquad v = \begin{bmatrix} 3 \\ 1 \end{bmatrix} s$$

with $s \neq 0$. To interpret these numbers as percentages, they have to add up to one. Hence

$$3s + s = 1, \qquad \text{or} \qquad s = 1/4.$$

Hence, the eigenvector is $v = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$ and the percentage of plants flowering is 75%.