MAT 1332, Winter 2014, Assignment 5
Due Friday March 7 by 3:00pm.
Late assignments will not be accepted; nor will unstapled assignments.
Professors in the math department will not lend you a stapler; do not ask for one.

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DGD (circle one): 1 2 3 4

Last Name ______________________________ First Name ______________________________

Student Number ______________________________(please write clearly)

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ______________________________

QUESTION 1. Find the matrix A such that

$$\left(3A^T - \begin{bmatrix} 1 & 3 & -2 \\ -4 & 5 & 1 \end{bmatrix}\right)^T = 3\begin{bmatrix} 7 & -5 & 4 \\ 9 & 12 & 3 \end{bmatrix}^T + \begin{bmatrix} -4 & 3 \\ 2 & 4 \\ -2 & 6 \end{bmatrix}$$

Answer

One has that:

$$\left(3A^T - \begin{bmatrix} 1 & 3 & -2 \\ -4 & 5 & 1 \end{bmatrix}\right)^T = 3\begin{bmatrix} 7 & -5 & 4 \\ 9 & 12 & 3 \end{bmatrix}^T + \begin{bmatrix} -4 & 3 \\ 2 & 4 \\ -2 & 6 \end{bmatrix} \Longleftrightarrow$$

$$(3A^T)^T - \begin{bmatrix} 1 & 3 & -2 \\ -4 & 5 & 1 \end{bmatrix}^T = 3\begin{bmatrix} 7 & 9 \\ -5 & 12 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} -4 & 3 \\ 2 & 4 \\ -2 & 6 \end{bmatrix} \Longleftrightarrow$$

$$3A = \begin{bmatrix} 1 & -4 \\ 3 & 5 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 21 & 27 \\ -15 & 36 \\ 12 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 3 \\ 2 & 4 \\ -2 & 6 \end{bmatrix} \Longleftrightarrow$$

$$A = \frac{1}{3}\begin{bmatrix} 18 & 26 \\ -10 & 45 \\ 8 & 16 \end{bmatrix} = \begin{bmatrix} 6 & 26/3 \\ -10/3 & 15 \\ 8/3 & 16/3 \end{bmatrix}.$$

QUESTION 2. Find the reduced row echelon form (using the Gauss elimination algorithm) of the matrix

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Answer

The reduced row echelon form is obtained as follows:

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - \frac{1}{2}R_1]{R_2 \to R_2 + \frac{1}{2}R_1} \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 3/2 & -3 & 3/2 \\ 0 & 0 & -3/2 & 0 & -3/2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - \frac{2}{3}R_2]{R_3 \to R_3 + R_2} \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 3/2 & -3 & 3/2 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix} \xrightarrow{R_4 \to R_4 + R_3} \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 3/2 & -3 & 3/2 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \to \frac{-1}{3}R_3}$$

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 3/2 & -3 & 3/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 3R_3} \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 3/2 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to \frac{2}{3}R_2} \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 + R_2} \begin{bmatrix} 2 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

QUESTION 3. Consider the system of equations with parameter a:

$$\begin{aligned} x+3y+9z &= 3 \\ 2x+7y+23z &= 2 \\ x+ay+a^2z &= a \end{aligned}$$

For which values of a does the system have (i) no solution; (ii) infinitely many solutions; (iii) a unique solution? Give the solution in cases (ii) and (iii).

Answer

Note that we have the following row operations:

$$\left[\begin{array}{ccc|c} 1 & 3 & 9 & 3 \\ 2 & 7 & 23 & 2 \\ 1 & a & a^2 & a \end{array}\right] \xrightarrow{\left[\begin{smallmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{smallmatrix}\right]} \left[\begin{array}{ccc|c} 1 & 3 & 9 & 3 \\ 0 & 1 & 5 & -4 \\ 0 & a-3 & a^2-9 & a-3 \end{array}\right] \xrightarrow{R_3 \to R_3-(a-3)R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 9 & 3 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & a^2-5a+6 & 5(a-3) \end{array}\right]$$

The next step is observe that: $a^2-5a+6=(a-2)(a-3)$.
(i) If no solution, then one has to impose the condition

$$a^2-5a+6=(a-2)(a-3)=0 \quad \text{AND } 5(a-3) \neq 0$$

$\Rightarrow \{a=2 \text{ OR } a=3\}$ AND $a \neq 3$. Hence $a=2$.

(ii) To get infinity many solutions, one needs at least one free variable. (In our case, only x_3 could be free, since x_1 et x_2 are independent.) We impose the condition:

$$a^2-5a+6=(a-2)(a-3)=0 \quad \text{AND } 5(a-3)=0 \Rightarrow \{a=2 \text{ OR } a=3\} \text{ AND } a=3.$$

We get that $a=3$. In this case sub $a=3$ in the last augmented matrix and continue the row reduction as follows:

$$\left[\begin{array}{ccc|c} 1 & 3 & 9 & 3 \\ 2 & 7 & 23 & 2 \\ 1 & 3 & 9 & 3 \end{array}\right] \xrightarrow{\cdots} \left[\begin{array}{ccc|c} 1 & 3 & 9 & 3 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - 3L_2} \left[\begin{array}{ccc|c} 1 & 0 & -6 & 15 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is the reduced row echelon form. The solutions are then obtained:

$$\left\{\begin{aligned} x_1 - 6x_3 &= 15 \\ x_2 + 5x_3 &= -4 \\ x_3 &\ \text{free} \end{aligned}\right. \iff \left\{\begin{aligned} x_1 &= 15 + 6x_3 \\ x_2 &= -4 - 5x_3 \\ x_3 &\ \text{free} \end{aligned}\right. .$$

(iii) To get a unique solution we impose the condition: $a^2 - 5a + 6 \neq 0 \Rightarrow a \neq 2$ AND $a \neq 3$. The reduced row echelon form is then:

$$\left[\begin{array}{ccc|c} 1 & 3 & 9 & 3 \\ 2 & 7 & 23 & 2 \\ 1 & a & a^2 & a \end{array}\right] \xrightarrow{\cdots} \left[\begin{array}{ccc|c} 1 & 3 & 9 & 3 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & a^2-5a+6 & 5(a-3) \end{array}\right] \xrightarrow{R_3 \to 1/(a^2-5a+6)R_3} \left[\begin{array}{ccc|c} 1 & 3 & 9 & 3 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 1 & 5/(a-2) \end{array}\right]$$

$$\xrightarrow{\left[\begin{smallmatrix} R_1 \to R_1 - 9R_3 \\ R_2 \to R_2 - 5R_3 \end{smallmatrix}\right]} \left[\begin{array}{ccc|c} 1 & 3 & 0 & (3a-51)/(a-2) \\ 0 & 1 & 0 & (-4a-13)/(a-2) \\ 0 & 0 & 1 & 5/(a-2) \end{array}\right] \xrightarrow{R_1 \to R_1 - 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & (15a-12)/(a-2) \\ 0 & 1 & 0 & (-4a-13)/(a-2) \\ 0 & 0 & 1 & 5/(a-2) \end{array}\right]$$

The solution is then:

$$\left\{\begin{array}{ccc} x_1 & = & (15a-12)/(a-2) \\ x_2 & = & (-4a-13)/(a-2) \\ x_3 & = & 5/(a-2) \end{array}\right.$$

QUESTION 4. (a) For each of the following matrices, calculate the determinant and the inverse (if it exists).

$$\text{(i)}\quad A = \begin{bmatrix} -1 & 17 \\ 4 & 2 \end{bmatrix} \qquad \text{(ii)}\quad B = \begin{bmatrix} 2 & -3 \\ -5 & 7.5 \end{bmatrix}$$

$$\text{(iii)}\quad M = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \qquad \text{(iv)}\quad N = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

Answer (use the back of pages or attach pages if you need more space to solve the questions, but please record the final answer here.)

(b) Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 3 \\ 2x + 5y + 3z &= 2 \\ x + 8z &= 4 \end{aligned}$$

Answer

(a)(i) We have that $\det(A) = (-1)(2) - (17)(4) = -70 \neq 0$. Thus A is invertible, and we have that

$$A^{-1} = \frac{1}{-70}\begin{bmatrix} 2 & -17 \\ -4 & -1 \end{bmatrix} = \begin{bmatrix} -1/35 & 17/70 \\ 2/35 & 1/70 \end{bmatrix}$$

(ii) Using the same formula for 2×2 determinants one has: $\det(A) = (2)(7.5) - (-3)(-5) = 0$. So A is NOT invertible.

(iii) One has that

$$\det(A) = 1\times 5\times 8 + 2\times 0\times 3 + 2\times 3\times 1 - (1)\times 5\times 3 - 0\times 3\times 1 - 2\times 2\times 8 = -1\,.$$

So A is invertible. To get its inverse form the extended augmented matrix: $[A|I_3]$. Then row reduce it as follows:

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\left[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1}\right]} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 + 2R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \xrightarrow{R_3 \to -R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \xrightarrow{\left[\substack{R_2 \to R_2 + 3R_3 \\ R_1 \to R_1 - 3R_3}\right]}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] = [I_3|A^{-1}]$$

thus

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}.$$

(iv) One has that

$$\det(A) = 1 \times 4 \times 5 + 2 \times 2 \times 4 + 6 \times (-1) \times (-1) - (-1) \times 4 \times 4 - 2 \times (-1) \times 1 - 2 \times 6 \times 5 = 0\,.$$

So A is not invertible.

(b)One has the following matricial equation:

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

and because A is invertible, one has:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -52 \\ 17 \\ 7 \end{bmatrix}$$

or in other words, the solution is given by:

$$\begin{cases} x_1 &= -52 \\ x_2 &= 17 \\ x_3 &= 7 \end{cases}$$