Concordia University Department of Computer Science
COMP 232 Mathematics for Computer Science

Fall 2014: Section R
Midterm 2
Solution to Part B

1. (4 marks) Let x be a real number and n be an integer. Fill in the blanks in the questions below:

   (a) If $x \notin Z$, then $\lceil x \rceil - \lfloor x \rfloor = 1$.

   (b) If $x \in Z$, then $\lceil x \rceil - \lfloor x \rfloor = 0$.

   (c) $\lceil x + n \rceil = n + \lceil x \rceil$.

   (d) $\lfloor n/2 \rfloor = (n-1)/2$ if n is odd.

2. (6 marks) Complete the proof of the statement: When n and k are positive integers, then

$$\lceil n/k \rceil = \lfloor (n-1)/k \rfloor + 1$$

.

Proof. Let $n = qk + r$ with $q, r \in \mathbb{Z}$ and $0 \le r \le k-1$. We study the following two cases, and show that in each case, LHS=RHS.

Case 1: $r = 0$. Then

$LHS = \lceil n/k \rceil = \lceil (qk+r)/k \rceil = q + \lceil r/k \rceil = q + 0 = q$

$RHS = \lfloor (n-1)/k \rfloor + 1 = \lfloor (qk + r - 1)/k + 1 \lfloor = \lfloor q - 1/k \rfloor + 1 = q - 1 + 1 = q$

Case 2: r lies in the range $1 \le r \le k-1$. Then

$LHS = \lceil n/k \rceil = \lceil (qk+r)/k \rceil = q + \lceil r/k \rceil = q + 1$

$RHS = \lfloor (n-1)/k \rfloor + 1 = \lfloor (qk + r - 1)/k + 1 \lfloor = q + \lfloor (r-1)/k \rfloor + 1 = q + 1$

In both cases, $LHS = RHS$, so $\lceil n/k \rceil = \lfloor (n-1)/k \rfloor + 1$.

3. (4 marks) Consider the following statement and the "proof" that follows it:

Statement: For all $a, b, c, m \in \mathbb{Z}$ with $m \ge 2$, if $ac \equiv bc \pmod{m}$ then $a \equiv b \pmod{m}$.

Proof: Suppose $a \equiv b \pmod{m}$. Then m divides $a - b$. This implies that m divides $ac - bc$. Therefore $ac \equiv bc \pmod{m}$.

The statement is false. Counterexample: $m = 5, a = 2, b = 3, c = 5$. Then clearly $a \equiv b \pmod{5}$ is false but $ac \equiv bc$ (mod) m.

4. (6 marks) Use mathematical induction to prove that 3 divides $n^3 + 2n$ for all non-negative integers n.

(a) (1 mark) First verify the base case.
$n = 0$. Then $n^3 + 2n = 0$. Since $3|0$, the basis ie verified.

(b) (1 mark) Write down the inductive hypothesis.
Suppose $3|(n^3 + 2n)$ for some non-negative integer n.

(c) (1 mark) Write down what you have to prove in the inductive step.
We want to prove that $3|((n+1)^3 + 2(n+1))$.

(d) (3 marks) Now carry out the proof.
$(n+1)^3 + 2(n+1) = n^3 + 3n^2 + 3n + 1 + 2n + 2$
$= (n^3 + 2n) + 3(n^2 + n + 1)$
Then $3|(n^3 + 2n)$ because of the indutive hyothesis, and clearly $3|3(n^2 + n + 1)$. Therefore $3|((n^3 + 2n) + 3(n+1)$ as needed.
By mathematical induction, 3 divides $n^3 + 2n$ for all non-negative integers n.