Functions in 3 ~~or~~ more variables

21-1

Linear approximation for $w = f(x,y,z)$ is

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

$$= L(x,y,z) \leftarrow \text{linearization of } f.$$

$$\Delta w = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z).$$

Differential

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz.$$

21-2

ex. The dimensions of a rectangular box are measured as 75, 60, 40 cm.

error ≤ 0.2 cm.

Find the maximum error for volume:

$$V = xyz$$

$$dV = yz\,dx + xz\,dy + xy\,dz.$$

$$|\Delta x|, |\Delta y|, |\Delta z| \leq 0.2$$

$$\Delta V \approx dV = (60)(40)(0.2) + (75)(40)(0.2) + (75)(60)(0.2) = 1980 \text{ cm}^3$$

error

The Chain Rule

$z = f(x,y)$ is differentiable

$x = g(t), \quad y = h(t)$

then $$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

If $z = x^2y + 3xy^4$ $\quad x = \sin 2t$, $y = \cos t$

find dz/dt when $t = 0$

$$\frac{dz}{dt} = (2xy + 3y^4)\, 2\cos 2t + (x^2 + 12xy^3)(-\sin t).$$

$$\left.\frac{dz}{dt}\right|_{t=0} = (0+3)(2\cos 0) + (0+0)(-\sin 0) = 6.$$

Let $z = f(x,y)$ be a differentiable function (21-4)

$x = g(s,t) \quad y = h(s,t)$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}.$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}.$$

Ex. If $z = e^x \sin y$, $x = st^2$ $\quad y = s^2t$

find $\partial z/\partial s$ and $\partial z/\partial t$.

(21-5)

$$\frac{\partial z}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st) =$$
$$= t^2 e^{st^2} \sin(s^2 t) + 2st \cos(s^2 t).$$

$$\frac{\partial z}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2) =$$
$$= 2st\, e^{st^2} \sin(s^2 t) + s^2 e^{st^2} \cos(s^2 t).$$

In general $\quad u(x_1, x_2, \dots, x_n)$

$\quad x_i(t_1, t_2, \dots, t_m)$

Then $$\frac{\partial u}{\partial t_j} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \dots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_j}.$$

Implicit Differentiation

(21-6)

Let $F(x,y) = 0$ determine $y = f(x)$ implicitly.

So that $F(x, f(x)) = 0$. Assume F is differentiable.

$$\frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

if $\frac{\partial F}{\partial y} \neq 0$, then $$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}.$$

1. Find y' if $x^3+y^3=6xy$ 21-7

$$F(x,y)=x^3+y^3-6xy=0.$$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{3x^2-6y}{3y^2-6x}=-\frac{x^2-2y}{y^2-2x}.$$

Suppose $F(x,y,z)=0$ $\quad z=f(x,y)$

Then $$\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{1}+\frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{0}+\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}=0$$

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}=0. \quad\Rightarrow\quad \text{if } \frac{\partial F}{\partial z}\neq 0,\ \frac{\partial z}{\partial x}=-\frac{F_x}{F_z}.$$

Similar for y: 21-8

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}.$$

Ex. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3+y^3+z^3+6xyz=1$.

$$F(x,y,z)=x^3+y^3+z^3+6xyz-1$$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{x^2+2yz}{z^2+2xy}.$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{y^2+2xz}{z^2+2xy}.$$

Directional derivatives.

$z = f(x,y)$

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Take $\vec{u} = (a, b)$, $|\vec{u}| = 1$. Then

$$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

is called the directional derivative of f at (x_0, y_0) in the direction of a unit vector u.