Ex. Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at $(1,1,3)$ 2011.

$f(x,y) = 2x^2 + y^2$

$f_x(x,y) = 4x$ $\qquad\qquad$ $f_y(x,y) = 2y$

$f_x(1,1) = 4$ $\qquad\qquad$ $f_y(1,1) = 2$

$$z - 3 = 4(x-1) + 2(y-1).$$

or

$$z = 4x + 2y - 3.$$

Linear Approximation

210-12

Ex. $f(x,y) = 2x^2 + y^2$

$L(x,y) = 4x + 2y - 3$ is the tangent plane.

$\uparrow$

approximation to f when (x,y) is near $(1,1)$.

$$f(x,y) \approx 4x + 2y - 3.$$

at point $f(1.1, 0.95) \approx 4(1.1) + 2(0.95) - 3 = 3.3.$

actual value of $f(1.1, 0.95) = 3.3225$

Given $z = f(x,y)$ 2013

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

is called the linearization of f at (a,b).

To Compute $L(x,y)$ means to provide a linear approximation to f.

If $z = f(x,y)$, then f is differentiable at (a,b) if Δz can be expressed in the form.

$$\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

If the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b) then f is differentiable at (a,b). 2014

Ex. $f(x,y) = x e^{xy}$

Show that it is differentiable at $(1,0)$ and find its linearization. Use it to appoximate $f(1.1, -0.1)$.

20-5

$f_x(x,y) = e^{xy} + xy e^{xy}$ $\qquad$ $f_y(x,y) = x^2 e^{xy}$

$f_x(1,0) = 1$ $\qquad$ $f_y(1,0) = 1.$

Both f_x and f_y are continuous, so f is differentiable

$$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) =$$

$$= 1 + 1(x-1) + 1 \cdot y = x+y$$

$$f(1.1, -0.1) \approx 1.1 - 0.1 = 1.$$

20-6

Differentials

$dy = f'(x)dx$ $\qquad$ if $y = f(x)$

or

$\Delta y \approx f'(x)\,\Delta x$

If $z = f(x,y)$ we define

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

So $f(x,y) \approx f(a,b) + dz$

linear approximation.

Ex. If $z = f(x,y) = x^2 + 3xy - y^2$
find the differential dz (20-7)

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (2x+3y)dx + (3x-2y)dy$$

If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz

$x = 2$, $dx = \Delta x = 0.05$, $y = 3$ $dy = \Delta y = -0.04$.

$$dz = (2(2) + 3(3))\, 0.05 + (3(2) - 2(3))(-0.04) = 0.65$$

$$\Delta z = f(2.05, 2.96) - f(2,3) = 0.6449.$$

Ex. The base radius and heigh of a right circular cone are measured as 10cm and 25cm. with a possible error as much as 0.1 cm in each. (20-8)

Use differential to estimate the maximum error in the calculated volume of the cone.

$$V = \pi r^2 h/3$$

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = \frac{2\pi r h}{3}dr + \frac{\pi r^2}{3}dh.$$

$|\Delta r| \le 0.1$ $|\Delta h| \le 0.1$

$\boxed{20\pi\, cm^3 \approx \approx 63 cm^3}$

Take $dr = 0.1$ $dh = 0.1$ $r = 10$ $h = 25$

$$dV = \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1) = 20\pi$$