Functions of three or More Variables (18-10)

$f(x,y,z)$

f is a rule that assigns to each ordered triple (x,y,z) in a domain $D \subseteq \mathbb{R}^3$ a unique real number denoted by $f(x,y,z)$.

Ex. Find the domain of f if

$$f(x,y,z) = \ln(z-y) + xy \cdot \sin z.$$

f is defined if $z-y>0$, so $D = \{(x,y,z) \mid z>y\}$.
is a half space.

Find the level surfaces for (18-2)

$$f(x,y,z) = x^2+y^2+z^2$$

The <u>level surface</u> consists of (x,y,z) such that $f(x,y,z)=k$ ($k \geq 0$ is in the range of f).

$x^2+y^2+z^2=k$ ← fixed gives a sphere of radius $\sqrt{k}$.

~~The~~ Similar for functions in 3 variables and more. (18-10)

If f is defined on a subset D of $\mathbb{R}^n$ then $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$ means that for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that

if $x \in D$ and $0 < |\vec{x} - \vec{a}| < \delta$ then $|f(\vec{x}) - L| < \varepsilon$.

f is continuous at $\vec{a}$ if

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a}).$$

(18-B)

A function of n variables is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of numbers. $\mathbb{R}^n$ denotes the set of all such n-tuples.

Ex. Profit function.

Company sells n products on the market.

"Price" of the first product is c_1

second c_2

n-th c_n.

What profit it makes after selling

18-B3.5

x_1 units of the first 'product'

x_2 second ...

x_n n-th

Answer $C = c_1 x_1 + c_2 x_2 + \ldots + c_n \cdot x_n$.

C is a function of $x_1, x_2, \ldots, x_n$.

Also can write $f(x) = \vec{c} \cdot \vec{x}$ dot product of vectors

$\vec{c} = (c_1, c_2, \ldots, c_n)$

$\vec{x} = (x_1, x_2, \ldots, x_n)$

So f is either a function of n real variable, of a point $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$, of a vector $\vec{x}$.

Several variables: 18-B

$$f_{x_i}(x_1, x_2, \ldots, x_n) = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots x_n) - f(x_1 \ldots x_n)}{h}$$

that is $g(x_i) = f(\cancel{x_1}, \ldots x_i, \ldots \cancel{x_n})$ constants

$g'(x_i) = f_{x_i}(x_1, \ldots x_n)$.

Higher derivatives

$(f_x)_x$, $(f_x)_y$, $(f_y)_x$, $(f_y)_y$, ...

Partial Differential Equations (19-7)

$z = u(x,y)$ satisfies

Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

Solutions are called harmonic functions

Ex $u(x,y) = e^x \sin y$ is a solution.

$u_x = \dfrac{\partial u}{\partial x} = e^x \sin y \qquad u_y = e^x \cos y$

$u_{xx} = e^x \sin y \qquad u_{yy} = -e^x \sin y.$

So $u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0.$

The wave equation (19-8)

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

a depends on the density/tension.

Ex. $u(x,t) = \sin(x - at)$ satisfies the wave equation

$u_x = \cos(x-at) \qquad u_t = -a\cos(x-at)$

$u_{xx} = -\sin(x-at) \qquad u_{tt} = -a^2 \sin(x-at) = a^2 u_{xx}.$

Tangent Planes & Linear Approximation. 14-9

S surface : $z = f(x,y)$

f has continuous first partial derivatives

$P = P(x_0, y_0, z_0) \in S$ a point.

C_1, C_2 are intersections with planes $y = y_0$ and $x = x_0$.

T_1, T_2 are tangent lines to C_1 and C_2 at P

Tangent plane at P is a plain containing both T_1 and T_2.

Plane passing through $P(x_0, y_0, z_0)$ has 14-10

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

$$a = -A/C \qquad b = -B/C$$

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

Intersection with $y = y_0 \Rightarrow z - z_0 = a(x - x_0)$

$x = x_0 \Rightarrow z - z_0 = b(y - y_0)$

a = slope of T_1 $= f_x(x_0, y_0)$

b = slope of T_2 $= f_y(x_0, y_0)$.