Limits and Continuity

Let f be a function of 2 variables whose domain D includes points close to (a,b).
Then we say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

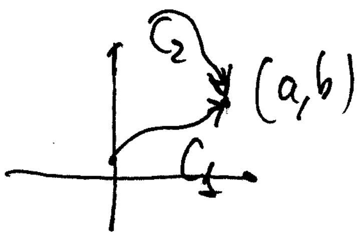
if for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that

distance

if $(x,y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$

then $|f(x,y) - L| < \varepsilon$.

distance.



If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and
$f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, where $L_1 \neq L_2$,

then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.

(18-3)

Ex. Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does NOT exist

$$f(x,y) = \frac{x^2-y^2}{x^2+y^2}$$

Approaches $(0,0)$ along Ox : $(x,0)\to(0,0)$

$x \to 0$

$$f(x,0) = \frac{x^2}{x^2} = 1.$$

—||— Oy : $(0,y) \to (0,0)$

$$f(0,y) = \frac{-y^2}{y^2} = -1$$

$\underline{1 \neq -1}$.

(18-4)

Ex If $f(x,y) = \dfrac{xy^2}{x^2+y^4}$ does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

approach along a line that passes through $(0,0)$.

$y = mx$ m is the slope.

$$f(x,y) = f(x,mx) = \frac{x(mx)^2}{x^2+(mx)^4} = \frac{m^2x^2}{x^2+m^4x^4} = \frac{m^2x}{1+m^4x^2}$$

So $f(x,y) \to 0$ as $(x,y)\to(0,0)$ along $y=mx$.

f has the same limit along every such line.

However, approach along $x = y^2$

$$f(x,y) = f(y^2,y) = \frac{y^2\cdot y^2}{(y^2)^2+y^4} = \frac{y^4}{2y^4} = \frac{1}{2} \neq 0.$$

Limit does not exist.

Ex Show that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ exists (18.5)

and compute it.

Let $\varepsilon>0$. Want to find δ s.t.

if $0<\sqrt{x^2+y^2}<\delta$, then $\frac{3x^2|y|}{x^2+y^2}<\varepsilon$.

But $x^2 \leq x^2+y^2$ since $y^2 \geq 0$ so $\frac{x^2}{x^2+y^2} \leq 1$

$\Rightarrow \quad \frac{3x^2|y|}{x^2+y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2+y^2}$

Set $\delta = \frac{\varepsilon}{3}$, then $\frac{3x^2|y|}{x^2+y^2} < \varepsilon$ for all (x,y) s.t. $0<\sqrt{x^2+y^2}<\delta$.

A function f of 2 variables is called continuous at (a,b) if (18.6)

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

f is continuous on D if f is continuous at every point (a,b) in D.

Ex. polynomial functions are continuous everywhere

rational functions are continuous in the domain.

Ex. Find $\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$

evaluate at $(1,2)$ $= 11$

Partial derivatives. (18-17)

$f(x,y)$ fix one of the variable $y=b$. (consider it as a constant).

$g(x) = f(x,b)$

$g'(x) = f'(x,b)$

replace b by y

$g'(x) = \frac{\partial}{\partial x} f(x,y)$.

We have $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ with respect to x and y.

$\frac{\partial f}{\partial x} = f_x(x,y)$ and $\frac{\partial f}{\partial y} = f_y(x,y)$.

Ex. If $f(x,y) = x^3 + x^2y^3 - 2y^2$ (18-8)

find $f_x(2,1)$ and $f_y(2,1)$.

Solution

$$f_x(x,y) = 3x^2 + 2xy^3$$

$$f_x(2,1) = 3\cdot 2^2 + 2\cdot 2\cdot 1^3 = 16.$$

$$f_y(x,y) = 3x^2y^2 - 4y$$

$$f_y(2,1) = 8.$$

Ex. $f(x,y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right)\cdot\frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right)\cdot\frac{1}{1+y}.$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right)\cdot\frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right)\cdot\frac{x}{(1+y)^2}.$$