Find the Maclaurin series for 15-1

$f(x) = \sin x$. $\quad f(0) = 0$.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$\sin' = \cos \qquad \cos(0) = 1$.

$\cos' = -\sin. \qquad -\sin(0) = 0$.

$-\sin' = -\cos \qquad -\cos(0) = -1$.

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$|f^{(n+1)}(x)| \le 1 \qquad M = 1$$

$$|R_n(x)| \le \frac{M}{(n+1)!} |x^{n+1}| \xrightarrow[n\to\infty]{} 0.$$

$$\cos x = (\sin x)' = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \right)' = \qquad 15\text{-}2$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{for all } x.$$

Ex. Find the Maclaurin series for the function $f(x) = x\cos x$

Just multiply: $x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$

Ex. Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\pi/3$. (15-3)

$$f(x) = \sin x \qquad f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$f'(x) = \cos x \qquad \frac{1}{2}$$

$$f''(x) = -\sin x \qquad -\frac{\sqrt{3}}{2}$$

$$f'''(x) = -\cos x \qquad -\frac{1}{2}.$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2 \cdot 1!}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2 \cdot 2!}\left(x - \frac{\pi}{3}\right)^2 - \frac{1}{2 \cdot 3!}\left(x - \frac{\pi}{3}\right)^3 + \dots$$

Ex. Find the Maclaurin ser for (15-4)

$f(x) = (1+x)^k$ k is any real number

$$f(x) = (1+x)^k$$

$$f'(x) = k(1+x)^{k-1}$$

$$\vdots$$

$$f^{(n)}(x) = k(k-1) \dots (k-n+1)(1+x)^{k-n}.$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1) \dots (k-n+1)}{n!} x^n.$$

Called binomial series

binomial coefficient $\binom{k}{n}$

Ratio Test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{k(k-1)\dots(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\dots(k-n+1)x^n}\right| =$$

$$= \frac{|k-n|}{n+1}|x| = \frac{\left|1-\frac{k}{n}\right|}{1+\frac{1}{n}}|x| \to |x| \text{ as } n\to\infty$$

Converges if $|x|<1$, diverges if $|x|>1$.

So if $|x|<1$ we have.

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n.$$

It converges at $x=1$ if $-1<k\le 0$ and converges at $x=\pm 1$ if $k\ge 0$.

Ex. Find the Maclaurin series for the function $f(x)=\frac{1}{\sqrt{4-x}}$ and its ratio of convergence

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}} \Rightarrow$$ binomial series with $k=-\frac{1}{2}$ and x replaced by $-\frac{x}{4}$.

So $$\frac{1}{\sqrt{4-x}} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n =$$

$$= \frac{1}{2}\left[1 + \frac{1}{8}x + \frac{1\cdot 3}{2!\,8^2}x^2 + \frac{1\cdot 3\cdot 5}{3!\cdot 8^3}x^3 + \dots + \frac{1\cdot 3\dots(2n-1)}{n!\cdot 8^n}x^n + \dots\right.$$

It converges if $|-x/4| < 1$ so $|x| < 4$.

$R = 4$

Ex. Find $\frac{1}{1\cdot 2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \dots$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n\cdot 2^n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{1}{2}\right)^n}{n}.$$

looks as for $\ln(1+x)$. $x - \frac{x^2}{2} + \frac{x^3}{3}$ with $x = \frac{1}{2}$.

$$\text{So} \quad = \ln\left(1+\tfrac{1}{2}\right) = \ln\tfrac{3}{2}.$$

Ex Evaluate $\int e^{-x^2}dx$ as infinite series.

$f(x) = e^{-x^2}$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$$

$$\int e^{-x^2}dx = C + x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \dots$$

$$\int_0^1 e^{-x^2}dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475$$

The Alternating series estimate gives that the error $< \frac{1}{11\cdot 5!} = \frac{1}{1320} < 0.001$

Ex Evaluate $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$ (15-9)

$$\frac{e^x - 1 - x}{x^2} = \frac{\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\right) - 1 - x}{x^2} =$$

$$= \frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \ldots$$

$$\lim_{x \to \infty} \left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!}\right) = \frac{1}{2}.$$

Ex. Find the first three nonzero terms in the Maclaurin series for $e^x \sin x$. 15-10.

$$= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\right)\left(x - \frac{x^3}{3!} + \ldots\right)$$

$$= x + x^2 + \frac{1}{3}x^3 + \ldots$$