$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Term-by-term differentiation and integration.

If $\sum c_n (x-a)^n$ has $R > 0$ then the function $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ is differentiable on $(a-R, a+R)$ and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$\int f(x)\,dx = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + \ldots =$$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radius of convergence stays the same.

Ex. Express $\frac{1}{1-x^2}$ as a power series. (14-2)

Find its radius of convergence.

We have $1 + x + x^2 + \ldots + x^4 + \ldots = \frac{1}{1-x}$

Differentiate: $1 + 2x + 3x^2 + \ldots = \frac{1}{(1-x)^2}$

So $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$ $\boxed{R = 1}$

Ex. Find a power series representation for $\ln(1+x)$ and its radius of convergence. (14-3)

$$\ln(1+x)' = \frac{1}{1+x}.$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 \qquad R=1.$$

Integrating $$\int \frac{1}{1+x}\,dx = \int (1-x+x^2-x^3+\dots)\,dx =$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + C.$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + C$$

Find C: Set $x=0$ (14-4)

Then $\ln(1+x) = C$ so $C=0$.

$R=1$

Ex. Find a power series representation for $f(x) = \tan^{-1} x$

$f'(x) = \dfrac{1}{1+x^2}$ so $\tan^{-1} x = \displaystyle\int \frac{1}{1+x^2}\,dx =$

$$= \int (1-x^2+x^4-x^6+\dots)\,dx = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Set $x=0$ $\tan^{-1} 0 = C = 0$. $\displaystyle \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$

Taylor and Maclaurin Series (14-5)

Which functions have power series representations? How to find them?

convergent

Suppose $f(x) = c_0 + c_1(x-a) + \dots$ $\quad |x-a|<R$

Let find c_i's the coefficients

$f'''(a) = 6c_3$

$f(a) = c_0$

$f^{(n)}(a) = n!\,c_n$

$f'(x) = c_1 + 2c_2(x-a) + \dots \qquad f'(a) = c_1$

$f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + \dots \qquad f''(a) = 2c_2.$

(14-6)

$$c_n = \frac{f^{(n)}(a)}{n!} \qquad \begin{pmatrix} 0! = 1 \\ f^{(0)} = f \end{pmatrix}$$

So we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \leftarrow$$

Taylor series of the function f at a. (about a, centered at a).

If $a=0$, then

$$f(x) = \sum_{n=0}^{n} \frac{f^{(n)}(0)}{n!}x^n \quad \leftarrow \text{Maclaurin series}$$

Ex. Find the Maclaurin series of $f(x) = e^x$ and radius of convergence (14-7)

$f'(x) = e^x \quad f^{(n)}(x) = e^x \quad f^{(n)}(0) = e^0 = 1$

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Find the radius:

Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{n+1} \to 0 < 1$.

So $R = \infty$ converges for all x.

When does the function equal to the sum of its Taylor series (assuming all derivatives exist)? (14-8)

Set $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$

nth degree Taylor polynomial of f at a.

If $\lim_{n\to\infty} T_n(x) = f(x)$, then $f(x)$ is the sum of the series.

Let $R_n(x) = f(x) - T_n(x)$. If $\lim_{n\to\infty} R_n(x) = 0$, then $f(x)$ is the sum.

remainder ~~(error)~~ of Taylor series.

14-9

<u>Fact</u>: If $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the nth degree Taylor polynomial of f at a and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a|<R$, then f is equal to the sum of its Taylor series on $|x-a|<R$.

14-10

How to show $\lim_{n\to\infty} R_n(x) = 0$:

Use: Taylor's Inequality

If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le R$,

then the remainder $R_n(x)$ of the Taylor series satisfies

$$|R_n(x)| \le \frac{M}{(n+1)!} |x-a|^{n+1} \quad \text{for } |x-a| \le R.$$

Ex Show that e^x is equal to the sum of its Maclaurin series (14-11)

$f(x) = e^x \qquad f^{(n+1)}(x) = e^x$

If R is any positive number and $|x| \leq R$

Then $|f^{(n+1)}(x)| = e^x \leq e^R$

So $(a = 0,\ M = e^R)$ (R is any, so $R = \infty$)

$$|R_n(x)| \leq \frac{e^R}{(n+1)!} \cdot |x|^{n+1}$$

But $\lim_{n \to \infty} \frac{e^R}{(n+1)!} |x|^{n+1} = 0$ (Ratio Test). So e^x is the sum.

Ex. Find the Taylor series for (14-12)

$f(x) = e^x$ at $a = 2$.

$f^{(n)}(2) = e^2$ so $\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n =$

$$= \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n.$$

Again R can be any, so the radius of convergence is ∞.