Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} C_n X^n = C_0 + C_1 X + C_2 X + \dots$$

$\underline{C_i\text{'s are constants}}$ called coefficients ; X is a variable.

$f(x) = \sum_{n=0}^{\infty} C_n X^n$ is the sum

Ex. Set $C_n = 1$ for all n, then

$f(x) = \sum_{n=0}^{\infty} C_n X^n = 1 + X + X^2 + \dots$ geometric series

$|x| < 1$ is convergent.

A series of the form

$$\sum_{n=0}^{\infty} C_n (X-a)^n = C_0 + C_1(X-a) + \dots$$

is called a power series about a

If $x = a$, then $\sum_{n=0}^{\infty} C_n (x-a)^n = C_0$ always converges.

Ex For what values of x $\sum_{n=0}^{\infty} n!\, x^n$ convergent?

Use Ratio Test. $a_n = n!\, X^n$

If $X \neq 0$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\, X^{n+1}}{n!\, X^n}\right| = \lim_{n\to\infty} (n+1)|X| = \infty$

$\Rightarrow$ diverges. | converges for $X = 0$.

Ex For what values of x

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} \text{ converge ?}$$

Set $a_n = (x-3)^n/n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right| = \frac{1}{1+\frac{1}{n}}|x-3| \to$$

$$\to |x-3| \text{ as } n \to \infty$$

So by the Ratio Test the series is absolutely convergent $\Rightarrow$ is convergent when $|x-3| < 1$ and divergent if $|x-3| > 1$.

Look at $x = 2, 4$

$x=4$ $\quad \sum \frac{1}{n}$ is divergent

$x=2$ $\quad \sum (-1)^n/n$ converges by the Alternating Test

So the power series is convergent for $2 \leq x < 4$ and diverges for other x.

Ex $\quad J(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ $\quad$ Bessel function of order 0. (13-5)

$a_n = (-1)^n x^{2n} / (2^{2n}(n!)^2)$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^{2n+2}}{2^{2n+2}(n+1)^2(n!)^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}} = \frac{x^2}{4(n+1)^2} \to 0 < 1$$

for all x

is convergent for all x.

Fact For a given power series (13-6)

$\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(i) The series converges only when $x=a$

(ii) The series converges for all x

(iii) There is a positive R s.t. the series converges if $|x-a|<R$ and diverges if $|x-a|>R$.

Case (i) $R=0$ $\quad$ Case (ii) $R=\infty$

R is called the radius of convergence (13-7)

Interval of converges consists of all values of x for which series converges

If x is an endpoint of the interval $-R<x<a+R$ anything can happend (converges/diverges).

Radius of convergence can be found using Ratio or Root Test

+ Endpoints use another Test if necessary.

Ex. Find the radius of convergence and interval of convergence of the series (13-8)

$$\sum_{n=0}^{\infty} \underbrace{\frac{(-3)^n x^n}{\sqrt{n+1}}}_{a_n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|-3x\sqrt{\frac{n+1}{n+2}}\right| = 3\sqrt{\frac{1+1/n}{1+2/n}}\,|x| \xrightarrow[n\to\infty]{} 3|x|$$

So by Ratio Test it converges if $3|x|<1$ and diverges if $3|x|>1$ $\Rightarrow R=\frac{1}{3}$

Endpoints $x = \pm \frac{1}{3}$ (13-9)

$x = -\frac{1}{3}$ $\quad \displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n\left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum \frac{1}{\sqrt{n+1}}$ ← diverges.

(~~use Integral test~~)
$p = \frac{1}{2}$ - series

$x = \frac{1}{3}$ $\quad \displaystyle\sum \frac{(-1)^n}{\sqrt{n+1}}$ converges by the Alternating Series Test

So the interval of convergence is

$$\left(-\frac{1}{3}, \frac{1}{3}\right]$$

Recall $\quad \displaystyle\frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n$ (13-10)

if $|x| < 1$ ← Geometric Series

Ex Express $\frac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence:

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Converges if $|-x^2| < 1$ and diverges if $|-x^2| \geq 1$

$\Rightarrow R = 1$. $\quad (-1, 1)$ = Interval of convergence

Ex Find a power series for $1/x+2$. 13-11

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2\left[1-\left(-\frac{x}{2}\right)\right]} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n =$$

$$= \sum \frac{(-1)^n}{2^{n+1}} x^n$$

$\left|-\frac{x}{2}\right| < 1$ converg

so $R = 2$ and $(-2, 2)$ int of conv.

Ex Find a power series of 13-8

$x^3/(x+2)$

$$\frac{x^3}{x+2} = x^3 \cdot \frac{1}{x+2} = x^3 \cdot \sum \frac{(-1)^n}{2^{n+1}} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3}$$