Absolute Convergence

Consider $\sum_{n=1}^{\infty} |a_n|$

A series $\sum a_n$ is called absolutely convergent if $\sum |a_n|$ is convergent

Ex. $\sum \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2}$ is absolutely convergent

Ex $\sum \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \dots$ is convergent but it is NOT absolutely convergent.

A series $\sum a_n$ is called <u>conditionally convergent</u> if it is convergent but <u>not</u> absolutely convergent

<u>Theorem</u> If a series $\sum a_n$ is absolutely convergent, then it is convergent

Indeed, $0 \le a_n + |a_n| \le 2|a_n|$

So $\sum |a_n|$ is converg $\Rightarrow 2\sum |a_n| = \sum 2|a_n|$ is converg

$\Rightarrow$ (by comparison) $\sum (a_n + |a_n|) = \sum a_n + \sum |a_n|$ is conv.

$\Rightarrow$ the difference $\sum (a_n + |a_n|) - \sum |a_n|$ is converg.

Ex. $\sum \frac{\cos n}{n^2}$ is conv or div. ? (12-3)

+ Is NOT alternating

$$\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right| = \sum \frac{|\cos n|}{n^2} \qquad |\cos n| \le 1 \ \forall n.$$

so $\frac{|\cos n|}{n^2} \le \frac{1}{n^2} \leftarrow$ 2-series is convergent

$\Rightarrow$ (by comparison) $\sum \left|\frac{\cos n}{n^2}\right|$ is convergent

$\Rightarrow$ $\sum \frac{\cos n}{n^2}$ is convergent

The Ratio Test (12-4)

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum a_n$ is absolutely convergent $\Rightarrow$ is convergent

(ii). If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, ~~then~~ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$

then $\sum a_n$ is divergent

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ no conclusion

Why (i) holds:

We compare with geometric series:

$L < 1$ so $L < r < 1$ for some r

Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ and $L < r$

There exists N s.t. $\forall n \geq N$ $\left|\frac{a_{n+1}}{a_n}\right| < r$.

or $|a_{n+1}| < |a_n| r$

So $|a_{N+i}| < |a_{N+i-1}| r < \dots |a_N| r^i$

$\sum_{k=1}^{\infty} \underbrace{|a_N| r^k}_{b_{N+k}}$ is convergent and $|a_{N+k}| < |b_{N+k}|$ $\forall k \geq 1$.

$\Rightarrow$ $\sum |a_n|$ is convergent.

Ex $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ abs. conv?

Use Ratio Test with $a_n = (-1)^n n^3/3^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(-1)^{n+1} (n+1)^3 / 3^{n+1}}{(-1)^n \frac{n^3}{3^n}} = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} =$$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \longrightarrow \frac{1}{3} \text{ as } n \to \infty$$

$\frac{1}{3} < 1$.

So it is absolutely convergent.

Ex. $\sum \frac{n^n}{n!}$ conv or div ? (12-7)

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)(n+1)^n}{(n+1)\cdot n!} \cdot \frac{n!}{n^n} =$$

$$= \frac{(n+1)^n}{n^n} = \left(1+\frac{1}{n}\right)^n \longrightarrow e \text{ as } n \to \infty$$

$$e > 1$$

is divergent.

The Root Test (12-8)

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum a_n$ is absolutely convergent

(ii) If $\lim \sqrt[n]{|a_n|} = L > 1$ or $\lim \sqrt[n]{|a_n|} = \infty$, then $\sum a_n$ is divergent

(ii) $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$ no conclusion.

Ex $\sum \left(\frac{2n+3}{3n+2}\right)^n$ 12-9

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{a_n} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \longrightarrow \frac{2}{3} < 1 \implies \text{converges.}$$

Rearrangements:

If $\sum a_n$ is absolutely convergent with $s = \sum a_n$

Then any rearrangements are convergent with the same sum s (Important: $\sum a_n$ has to be abs. conv)

How to test series on converg./diverg 12-10

1. If the series is $\sum \frac{1}{n^p}$ (p-series)

   $p > 1 \implies$ conv. $\quad p \leq 1 \implies$ diverg.

2. If it is $\sum c r^{n-1}$ (geometric series)

   $|r| < 1$ converg. $\quad |r| \geq 1$ diverges.

3. Similar to p-series or geom. ser then use one of the comparison tests

4. See if $\lim_{n\to\infty} a_n \neq 0$ (2-11)!

$\Rightarrow$ Divergent

5. If $\sum (-1)^{n-1} b_n$ then it is alternating $\Rightarrow$ Alternating series test.

6. If it involves factorials $n!$ or nth powers $\Rightarrow$ Ratio Test

7. If $a_n = (b_n)^n$ $\Rightarrow$ Root Test.

8. If $a_n = f(n)$ where $\int_1^\infty f(x)\,dx$ is easy $\Rightarrow$ Integral Test

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~~8~~. Ex $\sum \frac{n-1}{2n+1}$ (12-12)

$a_n \to \frac{1}{2} \neq 0$ Divergent

Ex $\sum \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ compare with p-series

$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{\frac{3}{2}}}{3n^3} = \frac{1}{3n^{3/2}}$ convergent. limit comparison test

Ex $\sum n e^{-n^2}$ $\int_1^\infty x e^{-x^2}\,dx$ $\Rightarrow$ Integral or Ratio Test.

Ex $\sum (-1)^n \frac{n^3}{n^4+1}$

Alternating Series Test

Ex $\sum \frac{2^k}{k!}$ Ratio Test

Ex $\sum \frac{1}{2+3^n}$ $\sim$ $\sum \frac{1}{3^n}$ Geometric Series

Comparison test