The comparison test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all $n \geq N$ then $\sum a_n$ is also convergent

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all $n \geq N$ then $\sum a_n$ is also divergent.

Ex. Is $\sum \frac{5}{2n^2+4n+3}$ convergent or divergent

We have $\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$

the largest term

$\sum \frac{5}{2n^2} = \frac{5}{2} \sum \frac{1}{n^2}$ ← 2-series, so is convergent

$\Rightarrow$ the original series converge by (i).

In the test it is enough to verify $a_n \leq b_n$ or $a_n \geq b_n$ for all $n \geq N$. (for some N)

Ex. $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ (11-3)

$\ln k > 1$ for $k \geq 3$ so $\frac{\ln k}{k} > \frac{1}{k}$, $k \geq 3$.

$\sum \frac{1}{k}$ ← 1-series so is divergent $(p=1)$

$\Rightarrow \sum \frac{\ln k}{k}$ is divergent by (ii) (here $N=3$).

The limit comparison test

Suppose $\sum a_n$, $\sum b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $c>0$.

then either both series converge or both diverge.

Ex. $\sum \frac{1}{2^n - 1}$ conv or divergent? (11-4)

Set $a_n = \frac{1}{2^n-1}$ and $b_n = \frac{1}{2^n}$.

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2^n-1)}{1/2^n} = \lim_{n\to\infty} \frac{1}{1-1/2^n} = 1 > 0$

But $\sum \frac{1}{2^n}$ converges, so is $\sum \frac{1}{2^n-1}$.

Ex $\sum \frac{2n^2+3n}{\sqrt{5+n^5}}$ converges or diverges (11-5)

$a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$, $b_n = \frac{2n^2}{n^{\frac{5}{2}}}$ leading terms

$\frac{a_n}{b_n} = \frac{2n^2+3n}{\sqrt{5+n^5}} \cdot \frac{n^{\frac{5}{2}}}{2n^2} = \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{5+n^5}} =$

$= \frac{2+\frac{3}{n}}{2\sqrt{\frac{5}{n^5}+1}}$ so $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{2}{2} = 1 > 0$.

$\sum b_n = \sum \frac{2}{n^{\frac{1}{2}}} = 2\sum \frac{1}{n^{\frac{1}{2}}}$ $\frac{1}{2}$ - series $\Rightarrow$ is divergent.

Recall $S = \sum_{k=1}^{\infty} a_k$ (11-6)

$S_n = \sum_{k=1}^{n} a_k$ $\quad S - S_n = R_n$

$t = \sum_{k=1}^{\infty} b_k$ $\quad t_n = \sum_{k=1}^{n} b_k$ $\quad t - t_n = T_n$

$\Big\}$ is the error (or remainder)

If $a_k \le b_k$ for $k \ge N$

then $R_k \le T_k$ for $k \ge N$.

So we can estimate the error R_k by T_k.

Ex. Use the sum of the first 100 terms to ... sum ... $\sum \frac{1}{n^3+1}$

Estimate the error.

(11-7)

$$\frac{1}{n^3+1} < \frac{1}{n^3} \leftarrow \text{3-series converges.}$$

so is $\sum \frac{1}{n^3+1}$ by the comparison test.

T_n for $\sum \frac{1}{n^3}$ can be estimated as

$$T_n \le \int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$$

So $R_n \le T_n \le \frac{1}{2n^2}$.

$n=100$: $R_{100} \le \frac{1}{2(100)^2} = 0.00005$

Alternating Series (11-8)

Series whose terms are alternately positive and negative.

$1 - \frac{1}{2} + \frac{1}{3} \ldots$ or $-1 + \frac{1}{2} - \frac{1}{3}$

Alternating Series Test

If the alternating series $\sum a_n$ satisfies

(i) $|a_{n+1}| \le |a_n|$ for all n

(ii) $\lim_{n\to\infty} |a_n| = 0$

then the series is convergent.

Ex $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ 11-9

satisfies

(i) : $\left|\frac{1}{n+1}\right| < \left|\frac{1}{n}\right|$

(ii) $\lim_{n\to\infty} \frac{1}{n} = 0$ so it converges.

Ex $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is alternating

$\lim_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4} \neq 0$ so the test can not be applied,

but by another test as $\frac{3}{4} \neq 0$ the series diverg.

Ex $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ conv or div? 11-10

(i) why $\frac{n^2}{n^3+1} > \frac{(n+1)^2}{(n+1)^3+1}$?

Use the derivative: Set $f(x) = \frac{x^2}{x^3+1}$

Then $f'(x) = \frac{x(2-x^3)}{(x^3+1)^2}$ and $f'(x) < 0$ if $2-x^3<0$ or $x > \sqrt[3]{2}$.

the latter means f is decreasing on $(\sqrt[3]{2}, \infty)$ (or for $n \geq 2$).

(ii) $\lim_{n\to\infty} \frac{n^2}{n^3+1} = 0$

So the series converge.

Alternating Series Estimation

(11-11)

If $s = \sum a_n$ is alternating that satisfies the alternating test then $|R_n| = |s - s_n| \le |a_{n+1}|$.

Ex. Find the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to 3 decimal places.

(11-12)

First, (i) $\frac{1}{(n+1)!} = \frac{1}{n!(n+1)} < \frac{1}{n!}$

(ii) $0 < \frac{1}{n!} < \frac{1}{n} \to 0$ so $\frac{1}{n!} \to 0$ as $n \to \infty$.

$$s = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots$$

Check that $|a_7| = \left|\frac{1}{7!}\right| = \frac{1}{5040} < \frac{1}{5000} = 0.0002$

and compute $s_6 \approx 0.368056$.

By estimation: $|s - s_6| \le b_7 < 0.0002$

does not affect 3rd decimal place

$\Rightarrow$ $s \approx 0.368$