10-1

Ex Is $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ divergent or convergent

$$= \sum_{n=1}^{\infty} (2^2)^n 3^{-(n-1)} = \sum \frac{4^n}{3^{n-1}} = \sum 4 \left(\frac{4}{3}\right)^{n-1}$$

$c = 4 \quad r = \frac{4}{3}$ since $r > 1 \Rightarrow$ diverges.

If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Ex. $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges | $\lim a_n = \lim \frac{n^2}{5n^2+4} = \frac{1}{5} \neq 0$.

10-2

<u>Facts</u> If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then so are the series $\sum c a_n$, $\sum (a_n + b_n)$, $\sum (a_n - b_n)$ and.

(i) $\sum c a_n = c \sum a_n$ $\qquad \left(\sum a_n := \lim_{n\to\infty} S_n.\right)$

(ii) $\sum (a_n + b_n) = \sum a_n + \sum b_n$

(iii) $\sum (a_n - b_n) = \sum a_n - \sum b_n$.

Ex. Find $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right)$. (10-3)

$\sum \frac{1}{2^n}$ is a geometric series with $c=\frac{1}{2}$ and $r=\frac{1}{2}$.

so $\sum \frac{1}{2^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

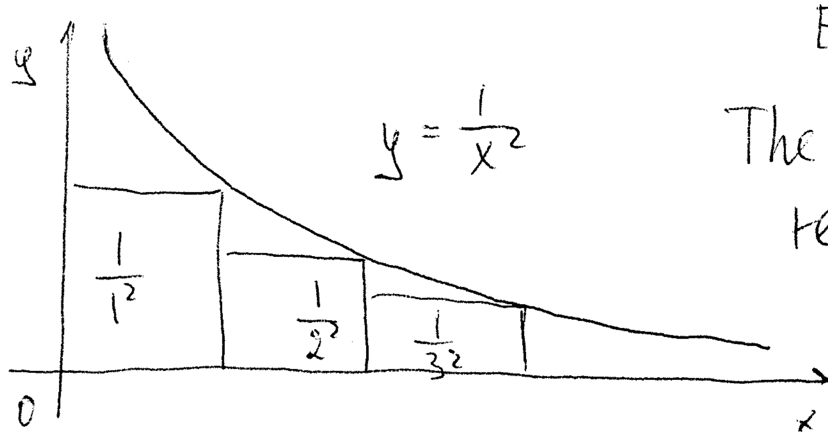
$$S_k = \sum_{n=1}^{k}\frac{1}{n(n+1)} = \sum_{n=1}^{k}\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1-\frac{1}{k}$$

$\lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(1-\frac{1}{k}\right) = 1.$ so $\sum \frac{1}{n(n+1)} = 1$

Hence $\sum\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right) = 3\cdot\sum\frac{1}{n(n+1)}+\sum\frac{1}{2^n} = 4.$

The integral test and estimates of Sums. (10-4)

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{1^2}+\frac{1}{2^2}+\dots$$



Exclute $\frac{1}{1^2}$ - area:

The total area of the remaining rectangles is smaller that the area under $y=\frac{1}{x^2}$

that is $\int_1^{\infty}\frac{1}{x^2}dx.$

But $\int_1^\infty \frac{1}{x^2}dx = 1$ So that

$$\frac{1}{1^2} + \int_1^\infty \frac{1}{x^2}dx = 2$$

So the partial sums are bounded it is also increasing, so the partial sums converge (Monotonic Sequence Theorem).

The Integral Test

Suppose f is a continuous, positive decreasing function on $[1, \infty)$ and let $a_n = f(n)$.

Then the series $\sum_{n=1}^\infty a_n$ is convergent if and only if the improper integral $\int_1^\infty f(x)dx$ is convergent.

We can also start from $n > 1$.

Important: f has to be decreasing starting from n.

Ex. Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ (10-7)

for convergence or divergence.

$f(x) = \frac{1}{x^2+1}$ is continuous, positive and decreasing on $[1, \infty)$ so

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \tan^{-1}x \Big]_1^t =$$

$$= \lim_{t\to\infty} \left(\tan^{-1}t - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$ ← convergent

So is the series.

Ex. Determine whether $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ (10-8)

converges or diverges

$f(x) = \frac{\ln x}{x}$ is positive and continuous for $x > 1$.

decreasing? $f'(x) = \frac{1/x \cdot x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$

So $f'(x) < 0$ for $\ln x > 1$ that is $x > e$

$\Rightarrow f$ is decreasing for $x > e$.

Apply the integral test $\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{\ln x}{x}\,dx =$

$= \lim \frac{(\ln x)^2}{2}\Big]_1^t = \infty$ is divergent

Ex For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent? 10-9

$\boxed{\text{called } p\text{-series}}$

If $p<0$, then $\lim_{n\to\infty} \frac{1}{n^p} = \infty$

If $p=0$ $\lim_{n\to\infty} \frac{1}{n^p} = 1$

$\Rightarrow$ It is divergent

If $p>0$, then $f(x)=\frac{1}{x^p}$ is continuous, positive and decreasing on $[1,\infty)$

$\int_1^\infty \frac{1}{x^p}dx$ converges if $p>1$ and diverges if $p\le 1$.

So is the series.

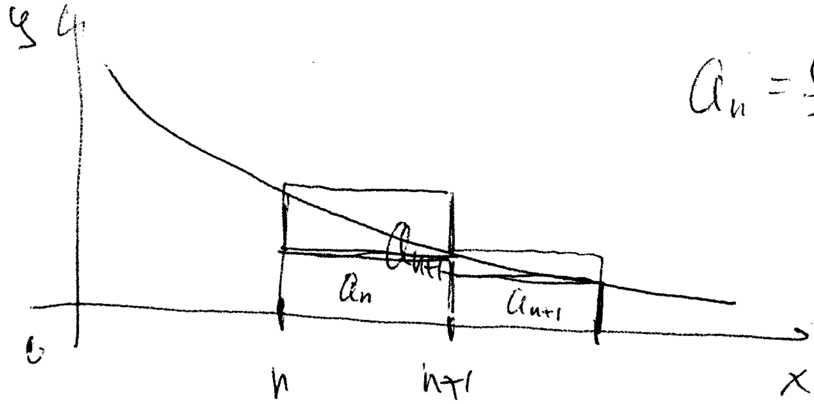
Estimating the sum of a Series 10-10

$S = a_1 + a_2 + \dots + a_n + \dots$

Comparing the areas on $[n,\infty)$

$$R_n = a_{n+1} + a_{n+2} + \dots + \dots \le \int_n^\infty f(x)\,dx.$$

$$R_n = a_{n+1} + a_{n+2} \ge \int_{n+1}^\infty f(x)\,dx$$



$a_n = f(n)$.

Here $R_n = S - S_n$ is the remainder (also called the error).

Suppose $f(k) = a_k$, f is continuous, positive, decreasing for $x \geq n$ and $s = \sum a_n$ converges

Let $R_n = s - s_n$

Then $\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$.

Ex. Apploximate the sum of the series $\sum \frac{1}{n^3}$ by using the sum of the first 10 terms. Estimate the error involved.

I. $f(x) = \frac{1}{x^3}$

$$\int_{n}^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty} -\frac{1}{2x^2}\Big]_n^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2} + \frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

II. $s_{10} = a_1 + \dots + a_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \dots + \frac{1}{10^3} \approx 1.1975$

III. $R_{10} \leq \int_{10}^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2\cdot 10^2} = \frac{1}{200}$ ← the size of the error is at most 0.005