Sequences (9-1)

list of numbers $a_1, a_2, \ldots, a_n, \ldots$

a_1: the first term; a_n: the nth term

Denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Ex 1. Find a formula for the general term a_n of the sequence

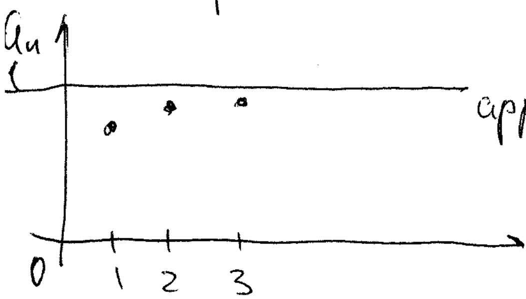
$\frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots$ $\qquad a_n = (-1)^{n-1} \cdot \frac{1}{n+1}$

2. $\left\{ \frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \ldots \right\}$ $\qquad a_n = (-1)^{n-1} \frac{n+2}{5^n}$

3. $\left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \right\}$ $\qquad a_n = \frac{n}{n+1}$

(9-2)

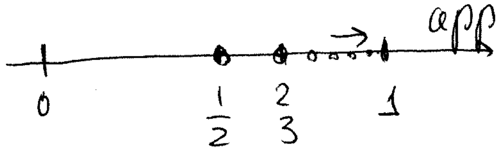
can draw the graph of the sequence $a_n = \frac{n}{n+1}$



approaches the line $a_n = 1$.

($\downarrow$ projection on Oa_n)

approaches 1.



$$\lim_{n \to \infty} \frac{n}{n+1} = 1.$$

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if $\lim_{n \to \infty} a_n$ exists (is a number) then the sequence <u>converges</u>

Otherwise, it <u>diverges</u>.

(9-3)

$\lim_{n\to\infty} a_n = L$ means that

for every $\varepsilon > 0$ there exists an integer N
such that for all $n > N$ we have

$$|a_n - L| < \varepsilon.$$

If a sequence is defined by means of a function $f(n) = a_n$ and $\lim_{x\to\infty} f(x) = L$,

Then $\lim_{n\to\infty} a_n = L$.

Ex. Since $\lim_{x\to\infty} \frac{1}{x} = 0$, $\lim_{n\to\infty} \frac{1}{n} = 0$.

(9-4)

We say that $\lim_{n\to\infty} a_n = \infty$ if
for every positive M there exists an integer N
such that for all $n > N$ $a_n > M$.

If $\lim_{n\to\infty} a_n = \infty$, we say that $\{a_n\}$ <u>diverges to ∞</u>

All limits laws, the squeeze theorem for functions hold for sequences: (assuming $x\to\infty$) replace x by n. and set $a_n = f(n)$.

Ex. Compute $\lim_{n\to\infty} \frac{\ln n}{n}$

We have $\lim_{x\to\infty} \frac{\ln x}{x} \underset{\text{l'Hospital Rule}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$

so $\lim_{n\to\infty} \frac{\ln n}{n} = 0$. (replaced x by n).

If $\lim_{n\to\infty} a_n = L$ and the function g is continuous at L, then

$\lim_{n\to\infty} g(a_n) = g(L)$.

Ex. Compute $\lim_{n\to\infty} \sin(\pi/n) = \lim_{x\to\infty} \sin(\pi/x)$.

$\sin$ is continuous at 0, so $\lim_{x\to\infty} \sin(\pi/x) = 0$.

$0 = \lim_{x\to\infty} \pi/x$.

The sequence $\{r^n\}$ is <u>convergent</u> if $-1 < r \le 1$

and is <u>divergent</u> for all other values of r

<u>Def.</u> A sequence $\{a_n\}$ is called <u>increasing</u> if $a_n < a_{n+1}$ for all n

that is $a_1 < a_2 < a_3 < \ldots$

It is called <u>decreasing</u> if $a_n > a_{n+1}$ for all n.

that is $a_1 > a_2 > \ldots$

It is monotonic if it is either increasing or decreasing.

Ex. The sequence $a_n = \frac{n}{n^2+1}$ is decreasing. (9-7)

Indeed, $\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$

Since
$$(n+1)(n^2+1) < n((n+1)^2+1).$$
$$n^3+n^2+n+1 < n^3+2n^2+2n.$$
$$1 < n^2+n. \quad \text{for all } n \geq 1.$$

Def. A sequence $\{a_n\}$ is <u>bounded above</u> if there is a number M s.t. $a_n \leq M$ for all n.
It is <u>bounded below</u> if there is a number m s.t. $m \leq a_n$ for all n.
If it is bounded above and below, it is <u>bounded</u>

<u>Important Fact</u> (Monotonic Sequence Theorem) (9-8)
Every bounded, monotonic sequence is convergent.

Ex. Investigate $\{a_n\}$ defined by the recurrence relation
$$a_1 = 2 \quad a_{n+1} = \frac{1}{2}(a_n+6) \quad \left(\begin{array}{l} a_2 = 4 \\ a_3 = 5 \ldots \end{array}\right).$$

It is increasing
$$a_{n+1} > a_n. \iff \frac{1}{2}(a_n+6) > \frac{1}{2}(a_{n-1}+6). \iff a_n > a_{n-1}$$
It is bounded above by 6.
$$a_n < 6 \iff \frac{1}{2}(a_{n-1}+6) < 6 \iff a_{n-1} < 6.$$
$\{a_n\}$ is convergent.

Let $\lim_{n\to\infty} a_n = L$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2}(a_{n-1}+6) = \frac{1}{2}\left(\lim_{n\to\infty} a_{n-1}\right) + 6$$

$$L = \frac{1}{2}L + 6 \quad \text{so} \quad \underline{L = 6}$$

Series

Let $\{a_n\}$ be a sequence.

Form a new sequence $\{S_n\}$.

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_n = a_1 + \dots + a_n$$

obtain the series of partial sums

If $\{S_n\}$ is convergent, then $\lim_{n\to\infty} S_n$ is called the sum of the series

If $\{S_n\}$ is divergent, then the series is called divergent

Ex Let $r, c \neq 0$ be fixed numbers.
Consider sequence $\{a_n\}_{n=1}^{\infty}$ defined by

$$a_n = c \cdot r^{n-1}$$

that is $\{c, c \cdot r, c \cdot r^2, \dots\}$.

(r is called the common ratio).

The series of partial sums $\{S_n\}$.

$$S_n = c + cr + cr^2 + \dots + cr^{n-1}$$

We have $r \cdot S_n = cr + cr^2 + \dots + cr^n$

So $S_n - r \cdot S_n = c - c \cdot r^n \Rightarrow S_n = \frac{c(1-r^n)}{1-r}$

The series S_n is called the geometric series.

If $-1 < r < 1$, $r^n \to 0$ as $n \to \infty$

so $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \frac{c(1-r^n)}{1-r} = \frac{c}{1-r} - \frac{c}{1-r} \lim\limits_{n\to\infty} r^n =$

$$= \frac{c}{1-r}.$$

The series $\{S_n\}$ converges to $\frac{c}{1-r}$ if $|r| < 1$.

If $r \leq -1$ or $r \geq 1$, it diverges (the limit does not exist).

If $r = 1$ it diverges to ∞.