Ex. Solve the differential eq. (8-1)

$$\frac{dy}{dx} = \frac{6x^2}{2y+\cos y}$$

$$\int (2y+\cos y)\,dy = \int 6x^2\,dx$$

$$y^2+\sin y = 2x^3+C \quad (*)$$

$(*)$ gives the general solution <u>implicitly</u>

Ex. Solve the equation (8-2)

$$y' = x^2 y$$

$$\frac{dy}{dx} = x^2 y$$

If $y \neq 0$ $\quad \dfrac{dy}{y} = x^2\,dx$

$$\int \frac{dy}{y} = \int x^2\,dx \qquad \ln|y| = \frac{x^3}{3}+C$$

$$|y| = e^{\ln|y|} = e^{\frac{x^3}{3}+C} = e^C \cdot e^{\frac{x^3}{3}}, \text{ so}$$

$$y = \pm e^C \cdot e^{\frac{x^3}{3}} \quad (\text{for } y \neq 0)$$

We check that $y = 0$ satisfies

$$y' = x^2 y$$

so it is also a solution.

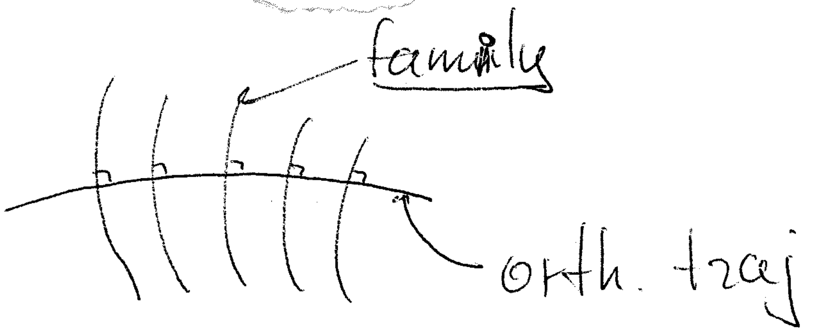
The general solution is then

$$y = A \cdot e^{\frac{x^3}{3}} \quad \text{for some } A. \ \left(\text{can be } 0\right)$$

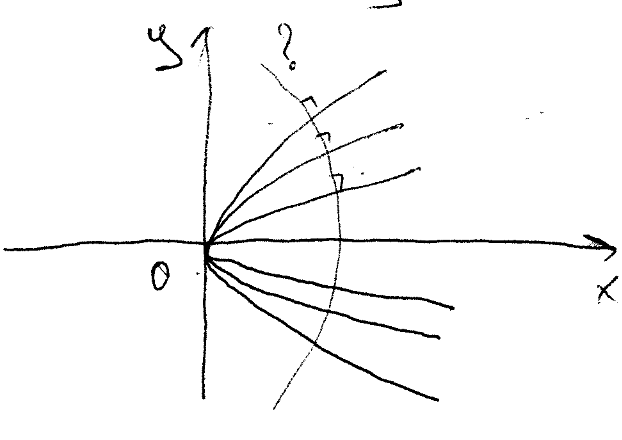
An orthogonal trajectory of a family of curves is a curve that intersects each curve of the family orthogonally.

8-4

Ex



Ex. Find the orthogonal trajectories of the family of curves $x = ky^2$, where k is arbitrary constant



I. Find a diff. eq. that is satisfied by all members of the family (8-5)

Differentiate : $x' = (ky^2)'$

$$1 = k2y \cdot y' \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{2ky}.$$

Eliminate constant k :

$$x = ky^2 \quad \text{so} \quad k = \frac{x}{y^2}$$

Hence, $\dfrac{dy}{dx} = \dfrac{1}{2 \cdot \frac{x}{y^2} \cdot y} = \dfrac{y}{2x}$.

the solutions are curves from the family.

$\Rightarrow$ slopes at (x,y) are $\dfrac{y}{2x}$.

II. On an orthogonal traj. the slope is $-\dfrac{1}{\text{slope}}$, so. the traj. have to satisfy. (8-6)

$$\frac{dy}{dx} = -\frac{2x}{y}$$

III Solving $\int y\,dy = -\int 2x\,dx$

$$\frac{y^2}{2} = -x^2 + C$$

$$x^2 + \frac{y^2}{2} = C.$$

the family of ellipses

Come back to the population growth (8-7)

$$\frac{dP}{dt} = kP \qquad P = P(t).$$

k — proportionality constant

Solve it

$$\int \frac{dP}{P} = \int k\,dt$$

$$\ln|P| = kt + C \qquad |P| = e^{kt+C} = e^C \cdot e^{kt}$$

$$\boxed{P = Ae^{kt} \qquad A = \pm e^C \text{ or } 0}$$

$$\boxed{P(0) = Ae^{k0} = A} \text{ ← is the initial population}$$

So the solution of the initial value problem (8-8)

$$P = P(t), \quad \frac{dP}{dt} = kP, \quad P(0) = P_0$$

is $P(t) = P_0 \cdot e^{kt}$.

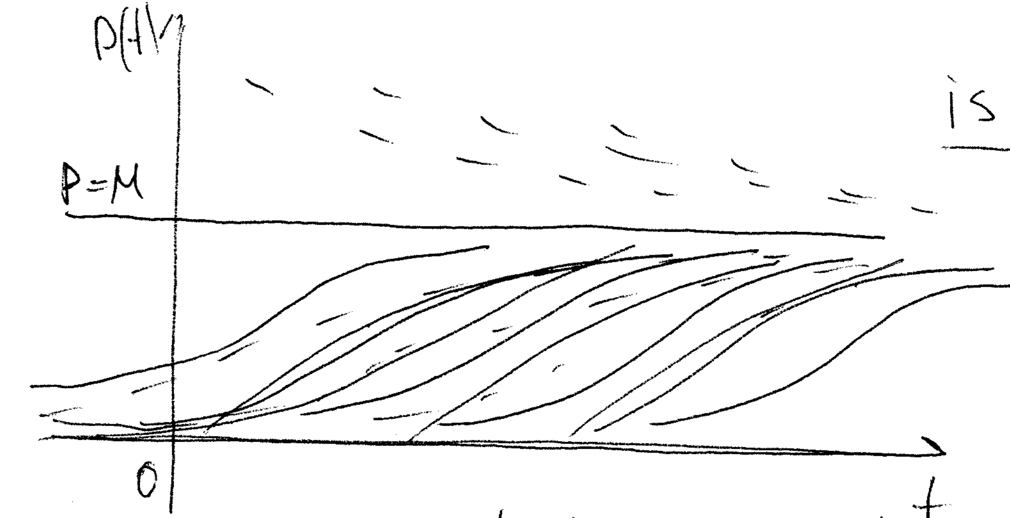
The logistic equation.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

M — carrying capacity.

Direction Field

8-9



is Autonomous

so any solution is a shift along Ot-line

P=M and P=0 are solutions

$$\int \frac{dP}{P(1-P/M)} = \int k\,dt$$

8-10

$$\frac{1}{P(1-P/M)} = \frac{M}{P(M-P)} = \frac{1}{P} + \frac{1}{M-P}.$$

$$\int\left(\frac{1}{P}+\frac{1}{M-P}\right)dP = \ln|P| - \ln|M-P| =$$

$$= \ln\left|\frac{M-P}{P}\right| + C_1 \quad \text{and}$$

$$\int k\,dt = kt \cancel{+} C_2$$

So $\ln\left|\frac{M-P}{P}\right| = -kt \cancel{+} C$

$$\Downarrow$$

$$\left|\frac{M-P}{P}\right| = e^{-c} e^{-kt} \Rightarrow \frac{M-P}{P} = Ae^{-kt},\ A = \pm e^{-c}$$

$$\Uparrow$$

$$\frac{M}{P} - 1 = Ae^{-kt}.$$

$$\Uparrow$$

$$P = \frac{M}{1+Ae^{-kt}}$$

Find A by putting $t=0$

$$P(0) = \frac{M}{1+A} \Rightarrow A = \frac{M-P(0)}{P(0)}$$

the initial population

So the solution to the logistic equation is

$$\boxed{P(t) = \frac{M}{1+Ae^{-kt}}, \quad \text{where } A = \frac{M-P(0)}{P(0)}}$$

Ex. Find the solution of the initial value problem

$$\frac{dP}{dt} = 0.08P\left(1-\frac{P}{1000}\right) \quad P(0)=100$$

and use it to find the population size $P(40)$.

$k=0.08 \quad M=1000 \quad P(0)=100$

So $P(t) = \dfrac{1000}{1+Ae^{-0.08t}}$, $A = \dfrac{1000-100}{100} = 9$.

$P(t) = \dfrac{1000}{1+9e^{-0.08t}}$ at $t=40$ $P(40) \approx 731.6$.