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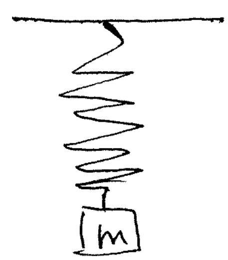
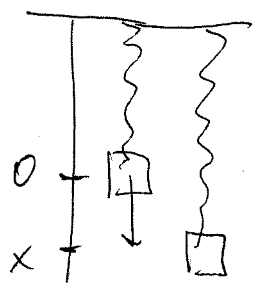
Ex. The Motion of a Spring



of mass m at the end of a vertical spring.

<u>Hooke's Law</u>: If the spring is streched (or compressed) x units from its natural length, then it exerts a force that is proportional to x:



restoring force $= -kx$ (k is the spring constant).

<u>Newton's Second Law</u>: (Force equals mass · acceleration)

$$m \cdot \frac{d^2x}{dt^2} = -kx$$ ← second order different. equation

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Differential Equation is an equation that contains an unknown function and one or more of its derivatives.

The order of a differential equation is the order of the highest derivative.

A function f is called a solution of a diff. equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation

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To solve = to find all solutions

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Sometimes we need to find the particular solution that satisfies a condition

$$y(t_0) = y_0.$$

This is called an initial condition.

an initial value problem

~~The problem~~

Ex. Show that $y = \dfrac{1+ce^t}{1-ce^t}$ satisfies $y' = \frac{1}{2}(y^2-1)$.

$\boxed{y' = \dfrac{2ce^t}{(1-ce^t)^2}}$

Find a solution of $y' = \frac{1}{2}(y^2-1)$ that satisfies the initial condition $y(0) = 2$.

$$2 = \frac{1+c}{1-c} \Rightarrow c = \frac{1}{3} \Rightarrow y = \frac{3+e^t}{3-e^t}$$

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Direction Fields

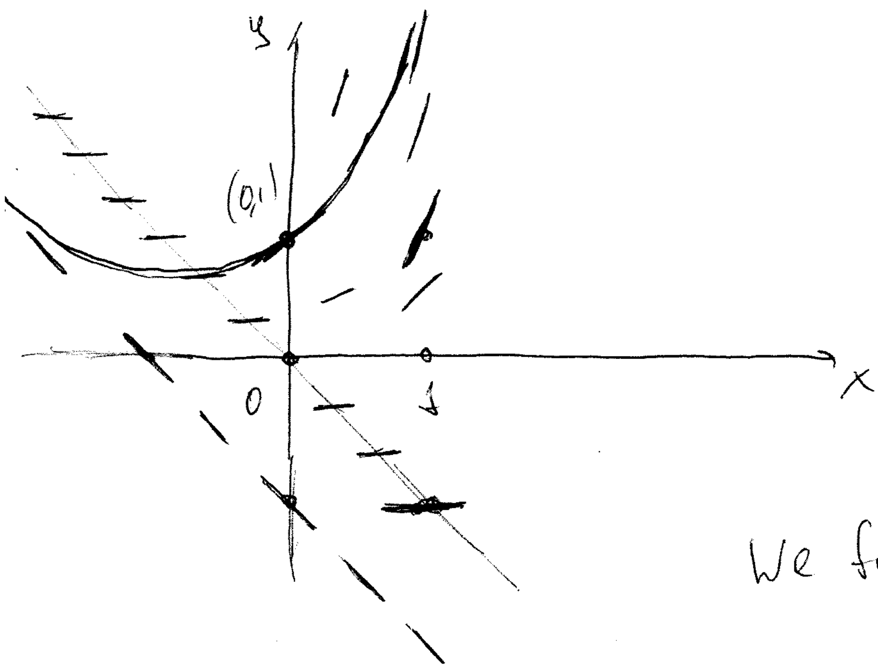
Want to sketch the graph of the solution of the initial value problem

$$y' = x + y \qquad y(0) = 1$$

$\hookrightarrow$ at the slope at any point (x,y) is $x+y$.

In particular at $(0,1)$ its slope is 1.

(7-5)



Draw short line segments at points (x,y) with slope $x+y$.

The result is called $\boxed{\text{a direction field}}$

We follow direction to sketch the solution

In general, suppose we have a first-order differential equation of the form (7-6)

$$y' = F(x,y)$$

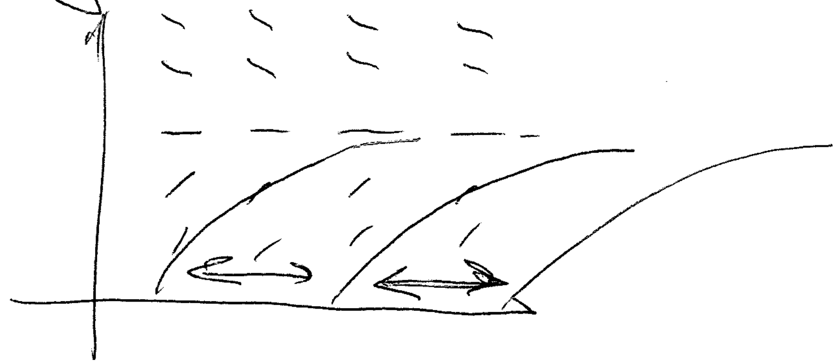
⇝ we draw its direction field. (or slope field).

⇝ draw the graph of solution for the initial value problem.

If $F(x,y)$ only depends on y but not on x then such a differential equation is called $\boxed{\text{autonomous}}$

(the slopes to two points with the same y-coord must be equal)

All solutions (grahs) of the autonomous equation are obtained by shifting along x-coordinate

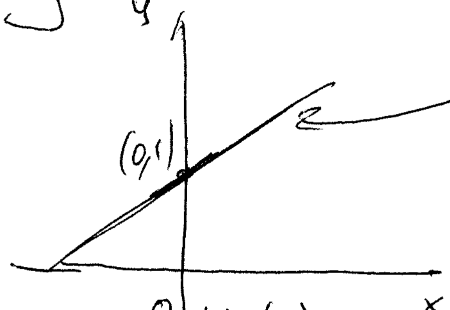


Euler's Method

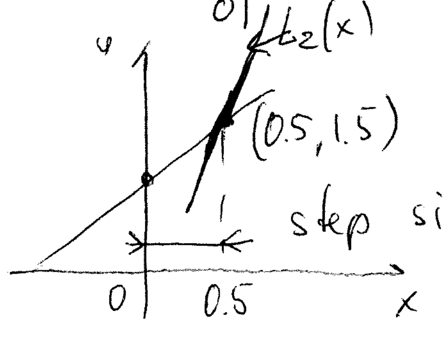
We want to approximate the solution by looking at its direction field

$y' = x + y \qquad y(0) = 1$

$L_1(x) = x + 1$ first approximation



Idea: To proceed a short distance along this line and then change the direction following by the direction field.
then repeat



step size is 0.5.

$y'(0.5) = 0.5 + 1.5 = 2$

So $L_2(x) = 1.5 + 2(x - 0.5)$ next approximation

In general $\quad y' = F(x,y)$
$\quad y(x_0) = y_0$

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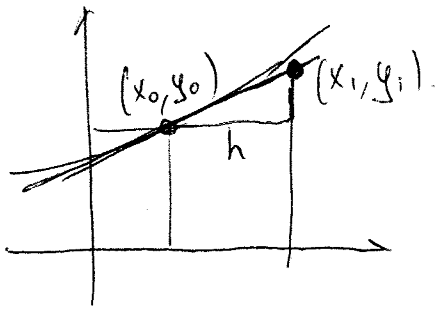
<u>h is the step size.</u>

Slope at (x_0, y_0) is $y' = F(x_0, y_0)$. $\rightsquigarrow \begin{cases} x_1 = x_0 + h \\ y_1 = y_0 + h \cdot F(x_0, y_0) \end{cases}$



$\rightsquigarrow \begin{cases} x_2 = x_0 + 2h \\ y_2 = y_1 + h \cdot F(x_1, y_1) \end{cases} \rightsquigarrow \cdots$

$$y_n = y_{n-1} + h\,F(x_{n-1}, y_{n-1})$$

are the approximate values for the solution of the initial value problem with step size h.

Ex. Using Euler's method with step size $h = 0.1$ construct a table of approximate values for the solution of the initial value problem

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$y' = x + y \qquad y(0) = 1$

$x_0 = 0 \qquad y_0 = 1 \qquad F(x,y) = x + y$

$y_1 = y_0 + h\,F(x_0, y_0) = 1.1$

$y_2 = 1.22$

$y_3 = 1.362$

n	x_n	y_n
1	0.1	1.1
2	0.2	1.22
3	0.3	1.362

Separable Equations

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A sep. eq. is a first-order diff. eq. in which the expression for $\frac{dy}{dx}$ can be factored as

$$\frac{dy}{dx} = \underbrace{g(x)}_{\text{function of } x} \cdot \underbrace{f(y)}_{\text{function of } y}.$$

if $f(y) \neq 0$ then

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

$$h(y) = \frac{1}{f(y)}.$$

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So $h(y)\,dy = g(x)\,dx$

$$\int h(y)\,dy = \int g(x)\,dx$$

defines y implicitly as a function of x.

Use the Chain Rule to justify it:

$$\frac{d}{dx}\left(\int h(y)\,dy\right) = \frac{d}{dx}\left(\int g(x)\,dx\right) = g(x)$$

$$\frac{d}{dx}\left(\int h(y)\,dy\right) = \frac{d}{dy}\left(\int h(y)\,dy\right) \cdot \frac{dy}{dx}$$

So $h(y)\frac{dy}{dx} = g(x)$

Ex. • Solve $\frac{dy}{dx} = \frac{x^2}{y^2}$ (7-13)

• Find the solution with $y(0) = 2$

• $y^2 dy = x^2 dx$

$\int y^2 dy = \int x^2 dx$

$\frac{1}{3} y^3 = \frac{1}{3} x^3 + C \quad \Rightarrow y = \sqrt[3]{x^3 + 3C} = \sqrt[3]{x^3 + \underset{3C}{\underset{\parallel}{K}}}$.

• $y(0) = \sqrt[3]{K} = 2 \quad \Rightarrow K = 8$

so $y = \sqrt[3]{x^3 + 8}$