Ex. Find the center of mass of a semicircular plate of radius r. (6-1)

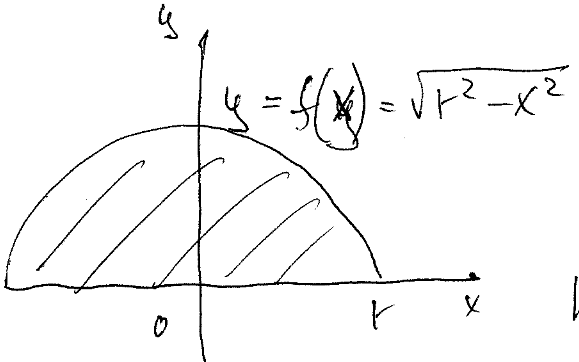
We recall that

$$\bar{x} = \frac{M_y}{m} = \frac{\rho \int_a^b x f(x)\,dx}{\rho \int_a^b f(x)\,dx} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\int_a^b \frac{1}{2}(f(x))^2\,dx}{\int_a^b f(x)\,dx}$$

is the area under $y = f(x)$

(6-2)



$a = -r, \quad b = r$

$\bar{x} = 0$ automatically.

We need only to find $\bar{y} = ?$

is the area of the semicircle

$$\int_a^b f(x)\,dx = \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = \frac{1}{2}\pi r^2$$

$$\int_a^b \frac{1}{2}[f(x)]^2\,dx = \frac{1}{2}\int_{-r}^{r} (r^2 - x^2)\,dx = \frac{1}{2}\left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} =$$

$$= \frac{2r^3}{3}.$$

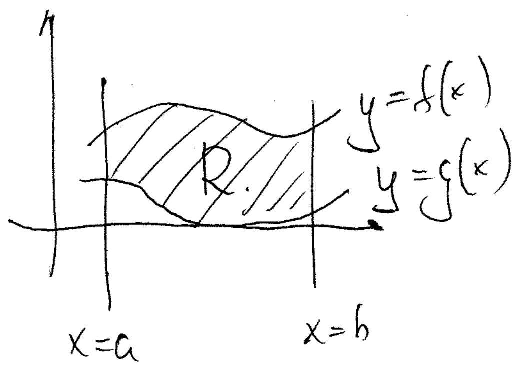
So $\bar{y} = \frac{2r^3}{3} \cdot \frac{2}{\pi r^2} = \frac{4r}{3\pi}$

6-3

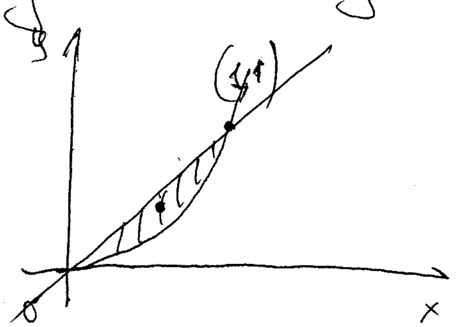
If the region R lies between two curves $y=f(x)$ and $y=g(x)$, where $f(x) \geq g(x)$, then

$$\bar{x} = \frac{1}{A}\int_a^b x\,(f(x)-g(x))\,dx \quad , \quad A = \int_a^b f(x)\,dx \ \left(\text{is the area}\right)$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[(f(x))^2-(g(x))^2\right]dx.$$



Ex. Find the centroid of the region bounded by the ~~line~~ $y=x$ and $y=x^2$ (6-4)



$$f(x)=x\ ,\ g(x)=x^2\ ,\ a=0,\ b=1$$

$$A = \int_0^1 (x-x^2)\,dx = \frac{x^2}{2}-\frac{x^3}{3}\Big]_0^1 = \frac{1}{6}.$$

$$\bar{x} = \frac{1}{A}\int_0^1 x[f(x)-g(x)]\,dx = \frac{1}{\frac{1}{6}}\int_0^1 x\,(x-x^2)\,dx = \frac{1}{2}.$$

$$\bar{y} = \frac{1}{A}\int_0^1 \frac{1}{2}\left[(f(x))^2-(g(x))^2\right]dx = \frac{1}{\frac{1}{6}}\int_0^1 \frac{1}{2}(x^2-x^4)\,dx = \frac{2}{5}.$$

So the centroid is $\left(\frac{1}{2}, \frac{2}{5}\right)$.

(6-5)

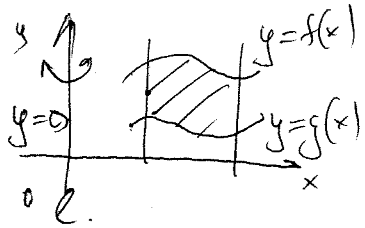
<u>Fact</u>: Let R be a plane region that lies entirely on one side of a line ℓ in the plane. If R is rotated about ℓ, then the volume of the resulting solid is

(the area A of R) $\cdot$ (the distance d traveled by the centroid of R).

Indeed, $V = \int_a^b 2\pi x\,[f(x)-g(x)]\,dx = 2\pi \int_a^b x\,[f(x)-g(x)]\,dx =$

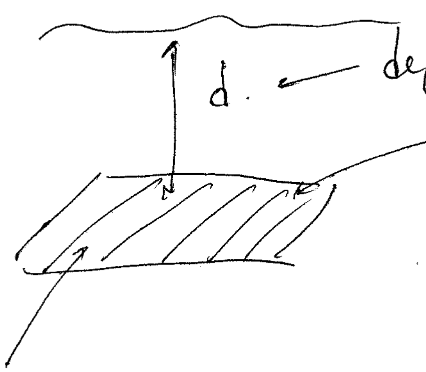
cylindrical shells

$= 2\pi(\bar{x}A) = (2\pi\bar{x})A = A\cdot d.$



<u>Pressure and Force</u>

(6-6)



A area of the plate.

The fluid above the plate has volume

$$\boxed{V = A\cdot d.}$$

So its mass is

$$\boxed{m = \rho\cdot V = \rho\cdot A\cdot d}$$

The force exerted by the fluid on the plate is

$$F = mg = \rho g A d.$$

gravity acceleration

Pressure P on the plate is defined to be (6-7)

$$P = \frac{F}{A} = \rho g d \quad \left(\text{newton}/m^2\right)$$
" pascal.

<u>Importan Principle</u>: At any point in a liquid the pressure is the same in all directions

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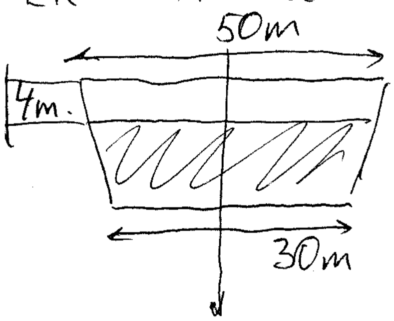
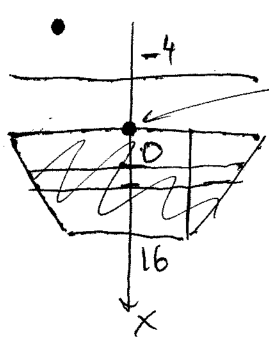
Ex. A dam has the shape of the trapezoid



Find the force on the dam due to hydrostatic pressure if the water level is 4m from the top of the dam

(6-8)



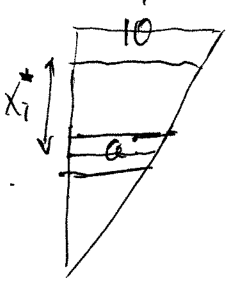
surface of water

Subdivide the interval $[0, 16]$ into n subintervals $[x_i, x_{i+1}]$

$x_i^* \in [x_i, x_{i+1}]$.

$\Delta x = x_{i+1} - x_i$ height of the strip

w_i the width of the strip



$$\frac{a}{16 - x_i^*} = \frac{10}{20}$$

$$a = \frac{16 - x_i^*}{2} = 8 - \frac{x_i^*}{2} \Rightarrow w_i = 2(15 + a) = 46 - x_i^*$$

A_i (the area) $\approx w_i \cdot \Delta x = (46 - x_i^*)\Delta x$.

$P_i \approx 1000 \cdot g \cdot x_i^*$ (by the formula)

$$F_i = P_i A_i \approx 1000 g x_i^* (46 - x_i^*) \Delta x.$$

$$F \approx \sum_{i=1}^{n} F_i \quad n \to \infty$$

$$F = \int_0^{16} 1000 g x (46 - x) dx \approx 4.43 \cdot 10^7 \ N.$$

Modeling with Differential Equations

Ex. <u>Population Growth</u>

Population grows at a rate proportional to the size of the population

t = time (variable)

$P(t)$ = the number of individuals in the population at the moment t

$\frac{dP}{dt}$ = the rate of growth

So $\frac{dP}{dt} = k \cdot P$, k is the proportionality constant

Here $P(t) > 0$

Possible solutions: (6-11)

Set $P(t) = Ce^{kt}$, then

$P'(t) = Cke^{kt} = k(Ce^{kt}) = kP(t)$

So Ce^{kt} is a solution for our differential equation.

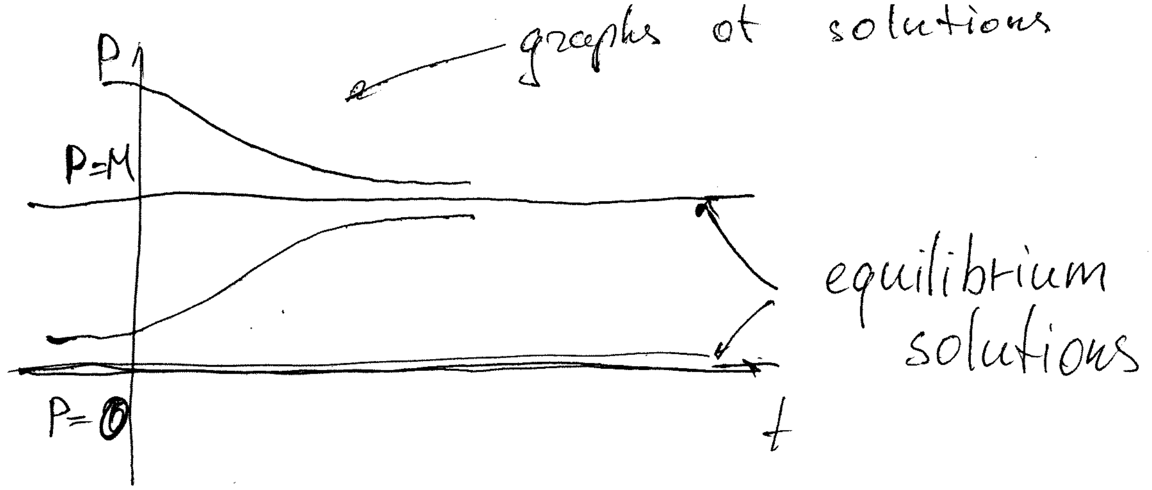
As $P(t) > 0$, $C > 0$

If $t = 0$ then $P(0) = Ce^{0} = C$, so C is the initial population.

What if the population levels off when it approaches its <u>carrying capacity</u> M (or decreases toward M if it ever exceeds M: (6-12)

- $\frac{dP}{dt} \approx k \cdot P$ if P is small (initial growth)
- $\frac{dP}{dt} < 0$ if $P > M$ (decreases if it exceeds M).

Then $\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)}$ ← logistic differential equation.


graphs of solutions
P
P=M
P=0
t
equilibrium solutions