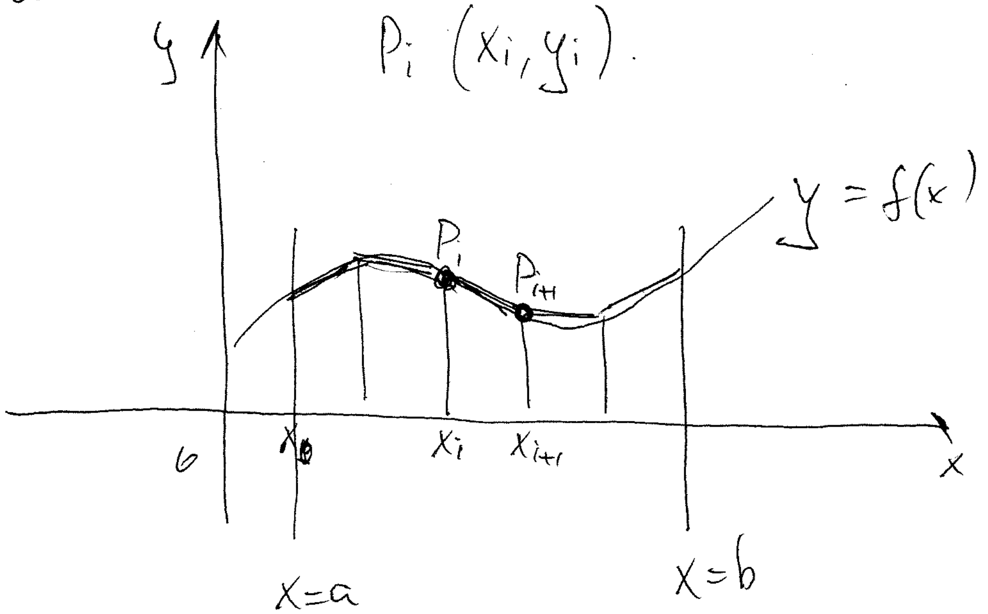
Arc Length

Suppose that a curve C is defined by the equation $y=f(x)$, f is continuous, $a \le x \le b$



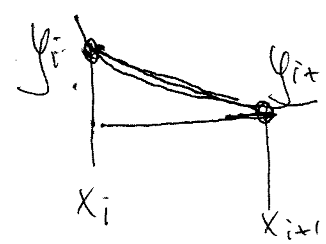
$$\Delta x = x_{i+1} - x_i$$

$$L_i = |P_i P_{i+1}|$$

$$L \approx \sum_{i=1}^{n} |P_{i-1} P_i|$$

Assume f has a continuous derivative $f'(x)$.

$$\Delta y_i = y_{i+1} - y_i$$



$$|P_i P_{i+1}| = \sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2} =$$

$$= \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

<u>Apply</u> the Mean Value Theorem to f on $[x_i, x_{i+1}]$

There exists $x_i^* \in [x_i, x_{i+1}]$ such that

$$y_{i+1} - y_i = f(x_{i+1}) - f(x_i) = f'(x_i^*)(x_{i+1}-x_i) = f'(x_i^*)\Delta x.$$

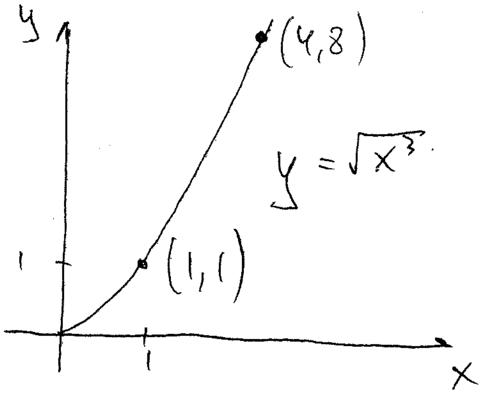
So $|P_i P_{i+1}| = \sqrt{(\Delta x)^2 + (f'(x_i^*)\Delta x)^2} = \sqrt{1+f'(x_i^*)^2} \cdot \Delta x.$

Now

$$L \approx \sum_{i=1}^{n} |P_{i-1}P_i| = \sum_{i=1}^{n} \sqrt{1+f'(x_{i-1}^{*})^2}\,\Delta x.$$

$$\boxed{L = \int_a^b \sqrt{1+f'(x)^2}\cdot dx.}$$

Ex. Find the length of the arc of the semicubical parabola $y^2=x^3$ between the points $(1,1)$ and $(4,8)$.



$$y = x^{3/2}$$

$$y' = \frac{3}{2}x^{\frac{1}{2}}$$

$$L = \int_1^4 \sqrt{1+\left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx =$$

$$= \int_1^4 \sqrt{1+\frac{9}{4}x}\,dx$$

$u = 1+\frac{9}{4}x \quad du = \frac{9}{4}dx.$

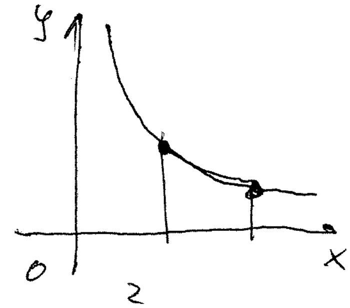
$x=1 \Rightarrow u=\frac{13}{4}$

$x=4 \Rightarrow u=10$

$$= \int_{\frac{13}{4}}^{10} \sqrt{u}\cdot\frac{4}{9}du = \frac{4}{9}\int_{\frac{13}{4}}^{10}\sqrt{u}\,du = \frac{4}{9}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big]_{\frac{13}{4}}^{10} =$$

$$= \frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)$$

Ex. (a) Set up an integral for the length of the arc of the hyperbola $xy = 1$ from $(1,1)$ to $(2, \frac{1}{2})$. 5-5



$$y = \frac{1}{x} \qquad y' = -\frac{1}{x^2}$$

$$L = \int_1^2 \sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\, dx = \int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx = \int_1^2 \frac{\sqrt{x^4+1}}{x^2}\, dx.$$

(b) Using Simpson's Rule with $a = 1$, $b = 2$, $n = 10$ and $\Delta x = 0.1$ and $f(x) = \sqrt{1 + 1/x^4}$ estimate the integral.

$$L \approx \frac{\Delta x}{3}\left[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + \dots + 2f(1.8) + 4f(1.9) + f(2)\right]$$
$$\approx 1.1321.$$

The arc length function. 5-6

$C : y = f(x) \qquad a \le x \le b.$

$s(x)$ = the distance along C from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$.

$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2}\, dt$$

By the fundamental theorem of calculus

$$s'(x) = \sqrt{1 + (f'(t))^2}$$

5-7

Ex. Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln x$ with $P_0\ (1,1)$ being the initial point

$$f'(x) = 2x - \frac{1}{8x}$$

$$1 + (f'(x))^2 = 1 + \left(2x - \frac{1}{8x}\right)^2 = \left(2x + \frac{1}{8x}\right)^2$$

$$\sqrt{1+(f'(x))^2} = 2x + \frac{1}{8x}$$ (take the absolute value of the root).

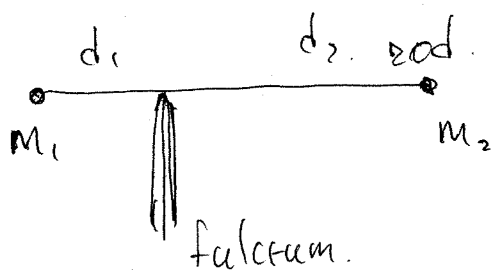
So $s(x) = \int_1^x \sqrt{1+(f'(t))^2}\,dt = \int_1^x \left(2t + \frac{1}{8t}\right)dt = t^2 + \frac{1}{8}\ln t \Big]_1^x =$

$= x^2 + \frac{1}{8}\ln x - 1$

Applications

5-8

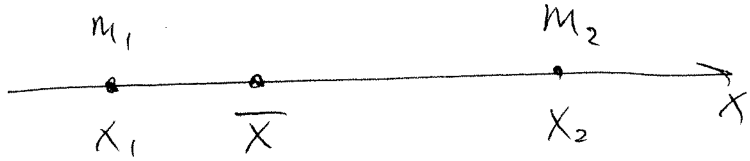
Moments and Centers of Mass



Archimedes

balance if $m_1 d_1 = m_2 d_2$.

Law of the Lever



$d_1 = \bar{x} - x_1$

$d_2 = x_2 - \bar{x}$

$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$

$m_1\bar{x} + m_2\bar{x} = m_1 x_1 + m_2 x_2$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

$m_1 x_1$, $m_2 x_2$ are called moments.

In general

If we have a system of n particles with masses $m_1, m_2, \dots, m_n$ located at $x_1, x_2, \dots, x_n$ on the line.

then
$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{m}$$
, where $m = m_1 + \dots + m_n$ is the total mass

$M = \sum_{i=1}^{n} m_i x_i$ is called the moment of the system about the origin.

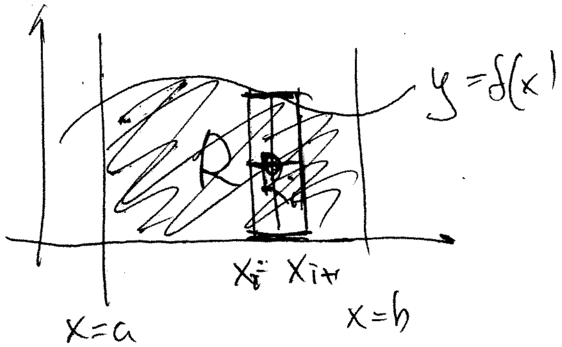
System of n particles on the plane

with masses $m_1, m_2, \dots, m_n$ located at $(x_1, y_1), \dots, (x_n, y_n)$

$M_y = \sum_{i=1}^{n} m_i x_i$ is the moment of the system about the y-axis.

$M_x = \sum_{i=1}^{n} m_i y_i$ —"— about the x-axis

The center of mass $\bar{x} = \frac{M_y}{m}$, $\bar{y} = \frac{M_x}{m}$.



(5-11)

Region: $y = f(x)$
$a \le x \le b$.

flat plate with uniform density ρ

Goal: to find the center of ~~its~~ mass of the plate (called the centroid of R).

Divide the interval into n subintervals.

$x_i^* \in [x_i, x_{i+1}]$ $\quad x_i^* = \frac{x_i + x_{i+1}}{2}$ the mid point

for R_i the centroid is $(\overline{x_i}, \frac{1}{2} f(\overline{x_i}))$.

the area is $f(\overline{x_i})\Delta x$ so its mass is $\rho f(\overline{x_i})\Delta x$.

(5-12)

$M_y(R_i) = (\rho f(\overline{x_i})\Delta x) \cdot \overline{x_i}$ = the moment about y-axis.

$= \rho \overline{x_i} f(\overline{x_i})\Delta x$

$M_y \approx \sum_{i=1}^{n} \rho \overline{x_i} f(\overline{x_i})\Delta x$

$$\boxed{M_y = \rho \int_a^b x f(x)\,dx}$$

Similarly,

$M_x(R_i) = (\rho f(\overline{x_i})\Delta x) \cdot \frac{1}{2} f(\overline{x_i}) = \rho \cdot \frac{1}{2} (f(\overline{x_i}))^2 \Delta x$.

$$M_x = \rho \int_a^b \frac{1}{2} (f(x))^2 dx.$$

$\overline{x} = \frac{M_y}{m}$, $\quad m = \rho A = \rho \int_a^b f(x)dx$.

$\overline{y} = \frac{M_x}{m}$