Improper Integrals

(4-1)

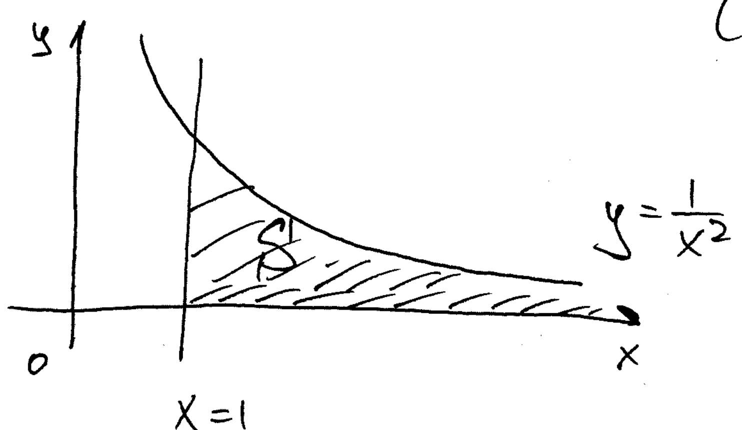
Goal: to extend the concept of a definite integral to the case where

- the interval is infinite, that is $[a, \infty)$ or $(-\infty, a]$
- f has infinite discontinuity in $[a,b]$

Such an extension is called an Improper Integral.

(4-2)

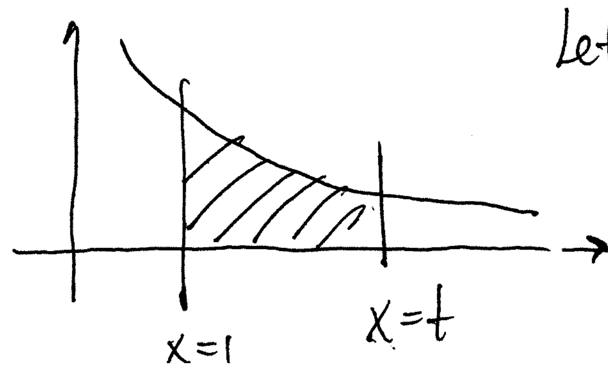
Type 1. Infinite intervals



Consider the region S bounded by $x = 1$, $y = \frac{1}{x^2}$ and $y = 0$.

Find the area of S.

Looks like its area is ∞, but it is NOT:



Let $A(t)$ be an area between $x = 1$ and $x = t$, $t \geq 1$.

$$A(t) = \int_1^t \frac{1}{x^2}\,dx =$$

$$= -\frac{1}{x}\Big]_1^t = 1 - \frac{1}{t}$$

Notice that $A(t) < 1$ always

and $A(t) \to 1$ as $t \to \infty$, indeed,

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1.$$

But the area of the region S is exactly

$$S = \lim_{t\to\infty} A(t) = 1.$$

Improper Integral of Type 1

We denote $\lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx$ by $\int_1^\infty \frac{1}{x^2}\,dx$

- If $\int_a^t f(x)\,dx$ exists for every $t \geq a$

  then $\int_a^\infty f(x)\,dx$ denotes $\lim_{t\to\infty}\int_a^t f(x)\,dx$.

  provided this limit exists.

- If $\int_t^b f(x)\,dx$ exists for every $t \leq b$

  then $\int_{-\infty}^b f(x)\,dx$ denotes $\lim_{t\to-\infty}\int_t^b f(x)\,dx$

  provided this limit exists

---

the expressions $\int_a^\infty \ldots$ and $\int_{-\infty}^b$ with infinities are called improper integrals of Type 1.

If those limits are finite numbers (4-5)
improper integrals are called

<u>convergent</u>

otherwise (are infinities) ...

<u>divergent</u>

If both $\int_a^\infty$ and $\int_{-\infty}^{a}$ are finite numbers
then we denote by $\int_{-\infty}^{\infty} f(x)dx$ its sum. that is

$$\int_{-\infty}^{\infty} f(x)dx := \int_{-\infty}^{a} f(x)dx + \int_a^\infty f(x)dx.$$

<u>Ex</u>. Determine whether $\int_1^\infty \frac{1}{x}dx$ is convergent or divergent: (4-6)

$$\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}dx = \lim_{t\to\infty} \ln|x|\Big]_1^t =$$

$$= \lim_{t\to\infty} \ln t - \ln 1 = \lim_{t\to\infty} \ln t = \infty$$

so it is divergent.

Ex Evaluate $\int_{-\infty}^{0} xe^x\,dx$

$$\int_{-\infty}^{0} xe^x dx = \lim_{t\to-\infty} \int_{t}^{0} xe^x dx =$$

integrate by parts $\underline{\underline{u=x}}$ $dv=e^x dx$. $du=dx$ $v=e^x$.

$\int xe^x dx$

$= \int x\,de^x = \int u\,dv = uv - \int v\,du = xe^x - \int e^x dx =$

$= xe^x - e^x$

$$= \lim_{t\to-\infty} xe^x - e^x \Big]_t^0 = \lim_{t\to-\infty}(-1 - te^t + e^t) =$$

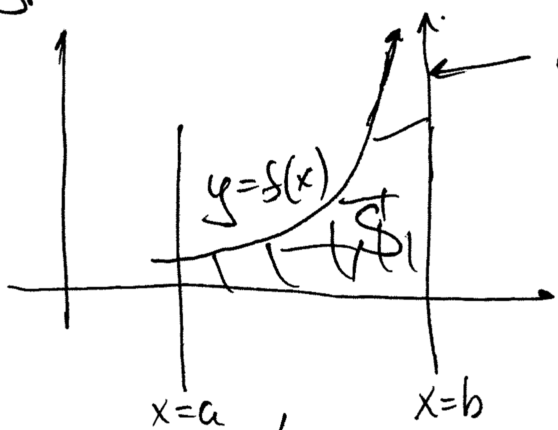
We know that $e^t \to 0$ as $t \to -\infty$

by l'Hospital's Rule

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty} \frac{t}{e^{-t}} = \lim_{t\to-\infty} \frac{1}{-e^{-t}} = \lim_{t\to-\infty}(-e^t) = 0.$$

So $\underline{\underline{= -1}}$.

Type 2. Dis continuous Integrals. (4-9)



asymptote.

The region is infinite in vertical direction

$$A(t) = \int_a^t f(x)\,dx$$

Improper integral of type 2.

We set $$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx.$$

the same as for the usual integral but f is <u>not</u> defined at b.

(4-10)

- If f is continuous on $[a, b)$ and is discontinuous at b then
$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$
if this limit exists.

- If f is continuous at $(a, b]$ and is discontinuous at a, then
$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$
if this limit exists

called an Improper Integral of type 2.

(4-11)

$\int_a^b f(x)\,dx$ is called <u>convergent</u> if the corresponding limit is a finite number

otherwise it is called <u>divergent</u>

---

if f has discontinuity at c, where $a<c<b$ and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

(4-12)

Ex. Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$

It is improper as $f(x) = \dfrac{1}{\sqrt{x-2}}$ has the vertical asymptote $x=2$.

(improper of type 2).

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+} 2\sqrt{x-2}\,\Big]_t^5 =$$

$$= \lim_{t\to 2^+} 2\left(\sqrt{3}-\sqrt{t-2}\right) = 2\sqrt{3}$$

Ex. Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible (4-13)

$x=1$ is a vertical asymptote of $\frac{1}{x-1}$

it occurs in the middle of $[0,3]$ so

$\int_0^3 = \int_0^1 + \int_1^3$

$\int_0^1 \frac{dx}{x-1} = \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1} =$

$= \lim_{t\to 1^-} \ln|x-1| \Big]_0^t =$

$= \lim_{t\to 1^-} \left(\ln|t-1| - \ln|-1|\right) = -\infty$

$\Rightarrow$ it is divergent.

A comparison test for improper integrals (4-14)

How to determine whether it is divergent or convergent without actually computing it:

Comparison Theorem.

Let f, g be continuous functions such that $f(x) \geq g(x) \geq 0$ for $x \geq a$

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent

Ex Show that $\int_0^\infty e^{-x^2}dx$ is convergent (4-15)

$$\int_0^\infty e^{-x^2}dx = \underbrace{\int_0^1 e^{-x^2}dx}_{\text{is a number}} + \int_1^\infty e^{-x^2}dx$$

$\underbrace{-x^2}_{g(x)} \le \underbrace{-x}_{f(x)}$ for all $x \ge 1$

But $\int_1^\infty e^{-x}dx = \lim_{t\to\infty} \int_1^t e^{-x}dx = -e^{-x}\Big]_1^t = e^{-1}$.

is convergent.

So is $\int_1^\infty g(x)\,dx \implies \int_0^\infty e^{-x^2}dx$ is convergent