Work = total amount of <u>effort</u> required to perform a task. (3-1)

<u>effort</u> = force in physics

if objects moves along a straight line with <u>position</u> $s(t)$, then the forse F is given by Newton's II Law of Motion

$$F = m\frac{d^2s}{dt^2}, \quad \text{where } m \text{ is its } \underline{\text{mass}}.$$

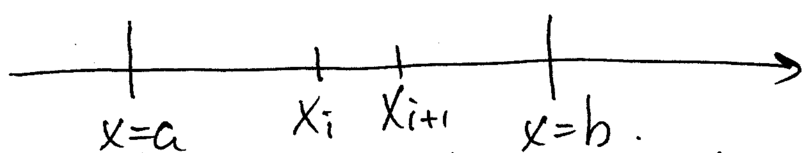
$(kg \cdot \frac{m}{s^2} = 1 \text{ Newton})$.

F is constan $\Leftrightarrow$ there is a constant acceleration

so $W = F \cdot d$. work = force $\times$ distance.
joule = newton $\cdot$ meter

What if the force is variable? (3-2)

Object moves along the x-axis in positive direction



$f(x)$ is the force at point $x \in [a,b]$.

<u>Subdivide</u> the interval into n parts

$\Delta x = x_{i+1} - x_i$ $\quad x_i^* \in [x_i, x_{i+1}]$. $\quad f(x_i^*)$ the force at x_i^*

$W_i \approx f(x_i^*)\Delta x$ is the work that is done moving from x_i to x_{i+1}

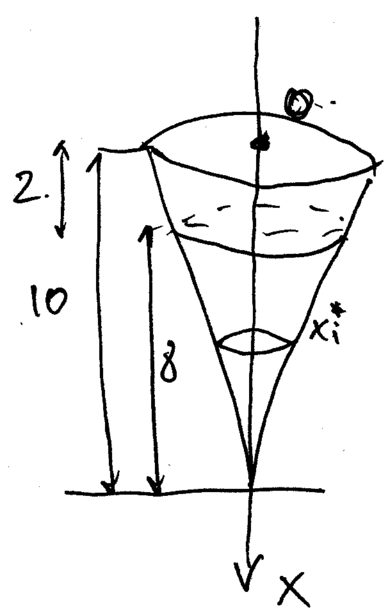
$$W \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$\underline{n \to \infty}$ (3-3)

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx.$$

Ex. A tank has the shape of an inverted circular cone with height 10m and base radius 4. It is ~~##~~ filled with water to a height of 8m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (density of water is 1000 kg/m³).

(3-4)



$[2, 10]$ divide into n subintervals

choose $x_i^* \in [x_i, x_{i+1}]$

i-th layer = circular cylinder with radius r_i and height Δx

$$\frac{r_i}{10 - x_i^*} = \frac{4}{10} \quad \text{using similar triangles}$$

$$r_i = \frac{2}{5}(10 - x_i^*).$$

$$V_i \approx \pi r_i^2 \Delta x = \frac{4\pi}{25}(10 - x_i^*)^2 \Delta x.$$

$$M_i = \text{density} \times \text{volume} \approx 1000 \cdot V_i = 160\pi (10 - x_i^*)^2 \Delta x$$

The force required to raise this layer must overcome the force of gravity so (3-5)

$$F_i = m \cdot g \approx (9.8) \cdot 160\pi (10 - x_i^*)^2 \Delta x = 1568\pi (10 - x_i^*)^2 \cdot \Delta x.$$

$$W_i \approx F_i \cdot x_i^* \approx 1568 \pi x_i^* (10 - x_i^*)^2 \Delta x.$$

$$W = 1568\pi \int_2^{10} x (10-x)^2 dx =$$

$$= 1568\pi \int_2^{10} (100x - 20x^2 + x^3) dx = 1568\pi \left(\frac{2048}{3}\right) \approx$$

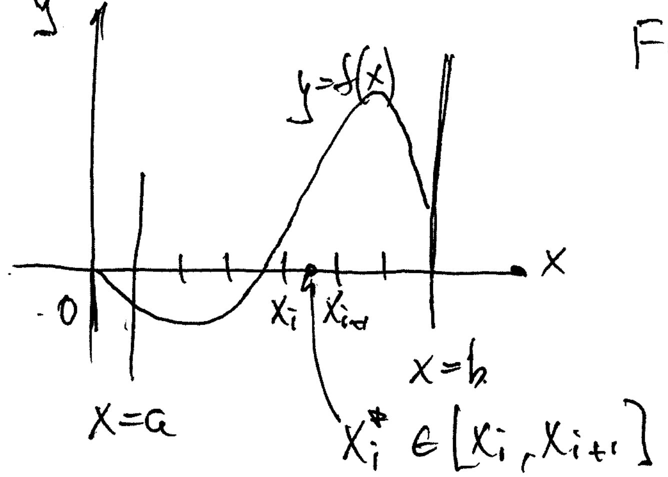
$$\approx 3.4 \times 10^6 \text{ J}.$$

Average Value of a Function (3-6)

At finitely many numbers

$$y_{av} = \frac{y_1 + y_2 + \dots + y_n}{n}.$$

What if we want to find an average temperature during a day (infinitely many numbers)?



Find the average value of $y = f(x)$ on $[a, b]$

Idea: Subdivide $[a,b]$ into n equal intervals, (3-7)

The length of each interval is $\Delta X = \frac{b-a}{n}$.

Choose points $x_i^* \in [x_i, x_{i+1}]$ for each interval.

$$\frac{f(x_1^*) + \dots + f(x_n^*)}{n} = \text{is the average}$$

$$= \frac{f(x_1^*) + \dots + f(x_n^*)}{\left(\frac{b-a}{\Delta X}\right)} = \frac{1}{b-a} \cdot [f(x_1^*) + \dots + f(x_n^*)] \Delta X$$

As $n \to \infty$ we obtain the average value of the funct. that is

$$\boxed{f_{av} = \frac{1}{b-a} \int_a^b f(x)\,dx.}$$

Ex. Find the average value of the function (3-8) $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

$a = -1$, $b = 2$

$$f_{av} = \frac{1}{b-a} \int_a^b f(x)\,dx = \frac{1}{2-(-1)} \int_{-1}^{2} (1+x^2)\,dx = 2.$$

So what is the average temperature?

$$T_{av} = \frac{1}{24} \int_0^{24} T(t)\,dt.$$

Is there a specific time when the temperature is the same as the average temperature? (3-9)

Or, generally, is there a number c at which $f_{av} = f(c)$

Yes, there is if f is continuous:

The Mean Value Theorem for Integrals.

If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that

$$f(c) = f_{av} = \frac{1}{b-a} \int_a^b f(x)dx$$

that is $$\int_a^b f(x)dx = f(c)(b-a)$$

Ex $f(x) = 1+x^2$ is continuous on $[-1,2]$ (3-10)

Then the M.V.T. says that there is $c \in [-1,2]$ such that

$$\int_{-1}^{2} (1+x^2)dx = f(c)(2-(-1))$$

So $$f(c) = f_{av} = 2$$

Hence $f(c) = 1+c^2 = 2$ so $c^2 = 1$, therefore $c = \pm 1$

There are <u>two</u> mean values at $c=-1$ and at $c=1$.

Ex. Show that the average velocity of a car over a time interval $[t_1, t_2]$ is the same as the average of its velocities during the trip.

$s(t)$ is the displacement of the car at time t.

$$\frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} \leftarrow \text{the average velocity}$$

$$v_{av} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v(t)\,dt = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} s'(t)\,dt = \frac{1}{t_2 - t_1}\, s(t)\Big]_{t_1}^{t_2}$$

average of its velocities

$$= \frac{\Delta s}{\Delta t}.$$