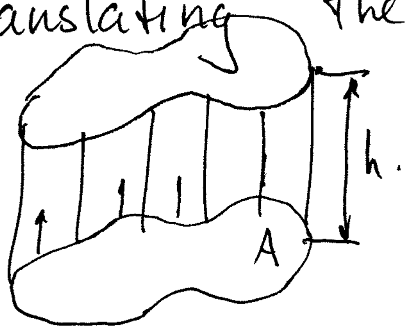
2-1

A. Cylinder is obtained by translating the area A along the direction perpendicular to A.



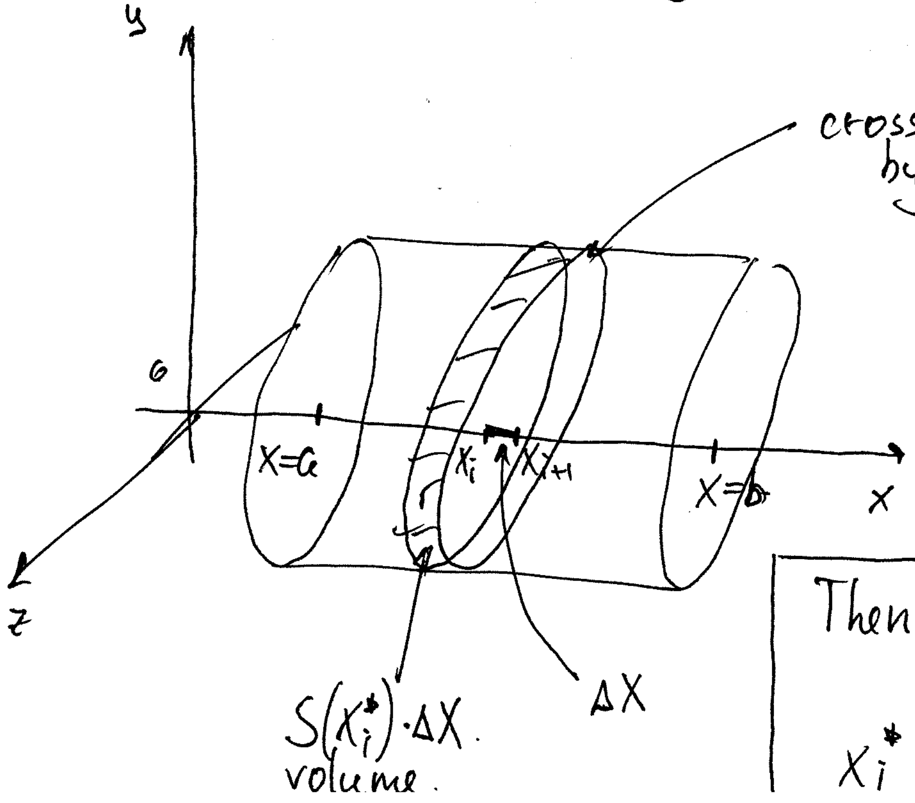
Its volume is given by

$$V = A \cdot h.$$

For instance, if A is a circle of radius r then its area is πr^2 and the volume of the respective cylinder is $\pi r^2 h$.

2-2

To find the volume V of a solid we cut it into cylindric pieces



Cross-section area by P_x a plane perpendicular to x-axis and passing through the point x. is $S(x)$.

Then $V = \sum_{i=1}^{n} S(x_i^*) \cdot \Delta x.$

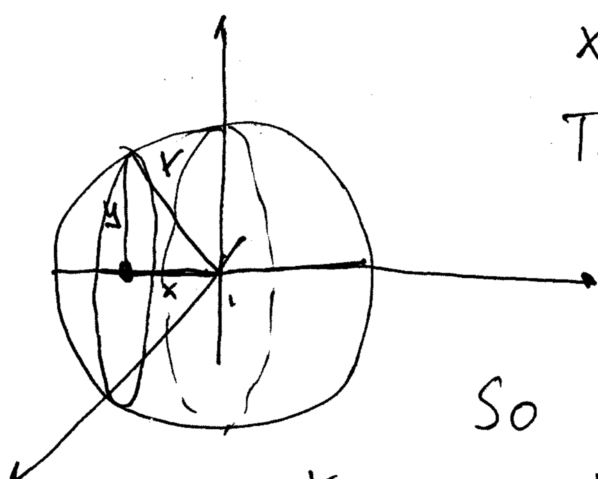
$x_i^* \in [x_i, x_{i+1}]$, $\Delta x = x_{i+1} - x_i$.

$$\boxed{\text{We obtain as } n \to \infty \quad V = \int_a^b S(x)\,dx \quad (x \text{ is a height}).}$$

Ex. For a cylinder, the cross-section area $S(x) = S$ is constant for all $x \in [a, b]$

So
$$V = \int_a^b S\,dx = S \cdot (b-a)$$

Ex: Volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$



$x = -r$, $x = r$

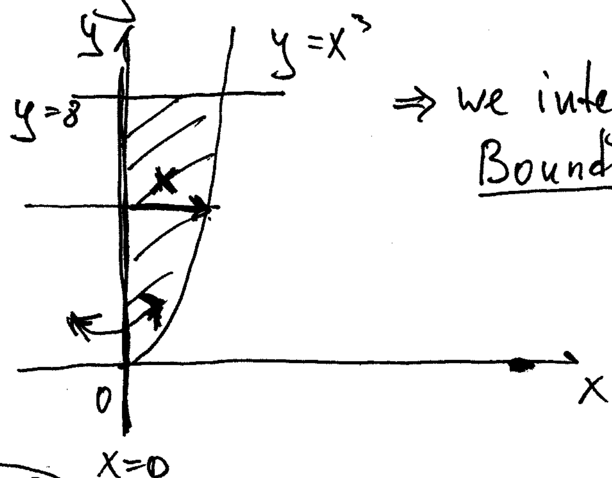
The cross-section at x is the circle with radius.

$$y = \sqrt{r^2 - x^2}$$

So $S(x) = \pi y^2 = \pi(r^2 - x^2)$.

Now
$$V = \int_{-r}^{r} S(x)dx = \int_{-r}^{r} \pi(r^2 - x^2)dx = \left(\pi r^2 x - \pi \frac{x^3}{3}\right)\Big]_{-r}^{r} =$$
$$= \frac{4}{3}\pi r^3$$

Ex.. Find the volume of the solid obtained by rotating the region bounded by $y=x^3$, $y=8$ and $x=0$ about the y-axis (2-5)



Slice the solid perpenducular to y-axis: $\Rightarrow$ we integrate along y

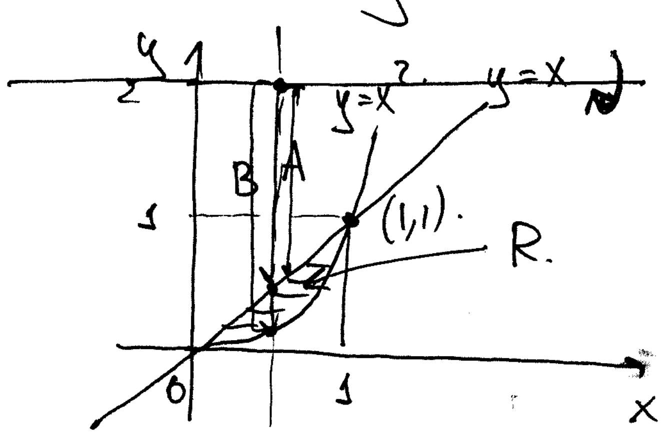
<u>Bounds</u>. $y=0$, $y=8$ (plains).

$x=\sqrt[3]{y}$ (the radius of the cylindrical piece).

$S(y)=\pi(\sqrt[3]{y})^2$ (the area of the cross-section).

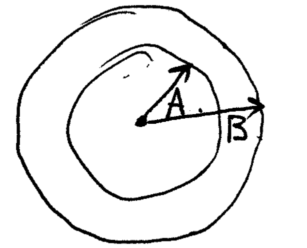
So, $V=\boxed{\displaystyle\int_0^8 \pi(\sqrt[3]{y})^2\cdot dy=\pi\frac{3}{5}y^{5/3}\Big]_0^8=\frac{96\pi}{5}}$

Ex.. The region R enclosed by the curves $y=x$ and $y=x^2$ is rotated about the line $y=2$. Find the volume of the resulting solid. (2-6)



<u>Bounds:</u> $x=0$, $x=1$.

Cross-section at x.



$A=2-x$, $B=2-x^2$

$S(x)=\pi B^2-\pi A^2=\pi\left((2-x^2)^2-(2-x)^2\right)=$

$=\pi(x^4-5x^2+4x)$

$$V = \int_0^1 S(x)\,dx = \int_0^1 \pi\left(x^4 - 5x^2 + 4x\right)dx =$$ 2-7

$$= \pi\left[\frac{x^5}{5} - 5\frac{x^3}{3} + 4\frac{x^2}{2}\right]_0^1 = \frac{8\pi}{15}$$

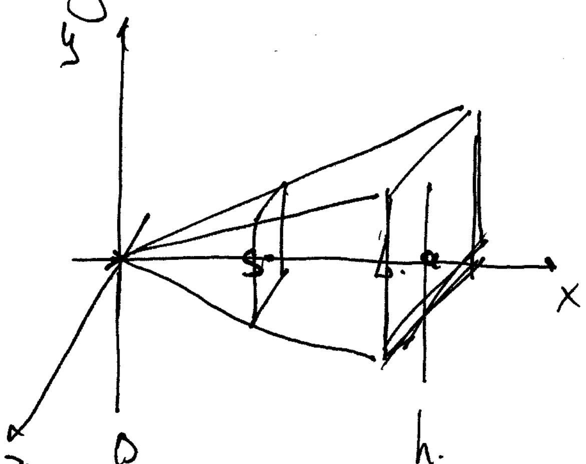
All these solids were obtained by revolving a region about a line.
They are called <u>solids of revolution</u>.

$$V = \int_a^b S(x)\,dx, \quad \text{where } S(x) = \pi\,(\text{radius})^2 \text{ or}$$

$$S(x) = \pi\left[(\text{outer radius})^2 - (\text{inner radius})^2\right]$$

<u>Ex</u>. Find the volume of a pyramid whose base is a square with side L and whose height is h. 2-8

$x = 0$, $x = h$.



$$\frac{x}{h} = \frac{s}{L} \quad \text{similarity}$$

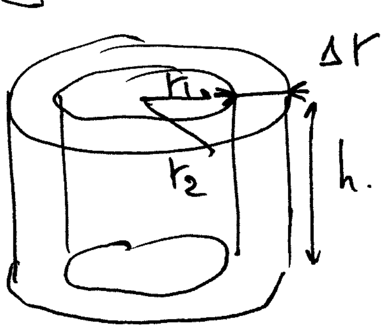
So $s = \dfrac{Lx}{h}$ and.

$$S(x) = \left(\frac{Lx}{h}\right)^2$$

$$V = \int_0^h \left(\frac{Lx}{h}\right)^2 dx = \frac{L^2 h}{3}$$

Volumes by Cylindrical Shells.

Cylindrical Shell:



$$V = V_2 - V_1 = \pi r_2^2 h - \pi r_1^2 h =$$

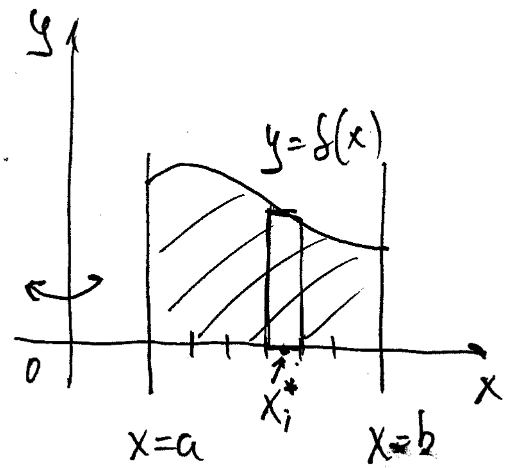
$$= \pi h \left(r_2^2 - r_1^2\right) =$$

$$= \pi h \left(r_2 - r_1\right)\left(r_1 + r_2\right) =$$

$$= 2\pi \frac{r_1 + r_2}{2} \cdot h \left(r_2 - r_1\right)$$

Set $r = \frac{r_1 + r_2}{2}$ (mid-point or average.) and $\Delta r = r_2 - r_1$

Then $V = \underbrace{2\pi r}_{\text{circumference}} \, \underbrace{h}_{\text{height}} \, \underbrace{\Delta r}_{\text{thickness.}}$



$$\boxed{\begin{array}{l} f(x) \geq 0, \quad x \in [a, b]. \\ 0 \leq a \leq b \end{array}}$$

Find the volume of the region revolving about $x = 0$.

Idea: Subdivide into intervals along OX (not OY).

$\Delta X = X_{i+1} - X_i$, $\overline{X_i} = \frac{X_i + X_{i+1}}{2}$ mid-point

height $= f(\overline{X_i})$, thickness $= \Delta X$,

circumference $2\pi \overline{X_i}$ $\Longrightarrow V_i = 2\pi \overline{X_i} \cdot f(\overline{X_i}) \cdot \Delta X$

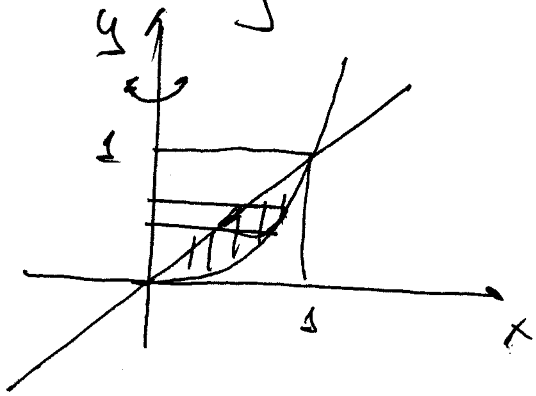
$$V \approx \sum_{i=1}^{n} 2\pi \overline{X_i} \cdot f(\overline{X_i}) \cdot \Delta X.$$

$$V = \int_a^b 2\pi x\, f(x)\,dx = 2\pi \int_a^b x f(x)\,dx$$

(x is a radius).

2-11

Ex. Find the volume of the solid obtained by rotating about y-axis the region between $y = x$ and $y = x^2$



$y = 0$, $y = 1$.

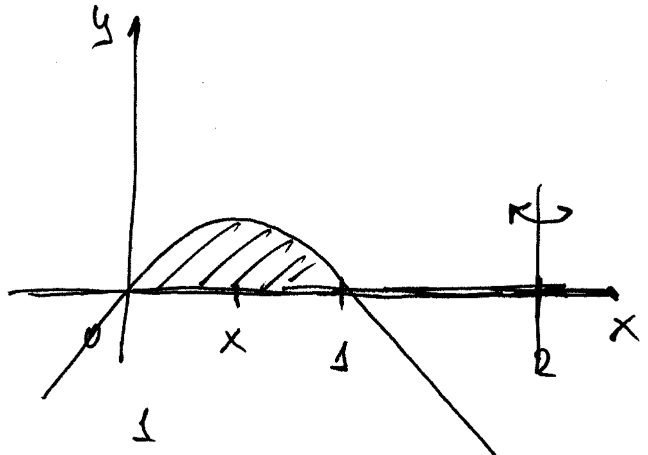
has radius x $\Rightarrow 2\pi x$.

height $x - x^2$

$$V = \int_0^1 2\pi x\,(x - x^2)\,dx = \frac{\pi}{6}.$$

2-12

Ex. Solid is obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.



radius $2 - x$ $\Rightarrow 2\pi(2 - x)$. is circumference.

height $x - x^2$

So

$$V = \int_0^1 2\pi(2 - x)(x - x^2)\,dx = 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\,dx =$$

$$= 2\pi \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 = \frac{\pi}{2}.$$