Diag F12 Solutions P

1. The distance from the point (1, 0, −1) to the plane with equation $2x - y + z = 7$ is:

A. $\sqrt{6}$ Note that $(0,0,7) \in P$

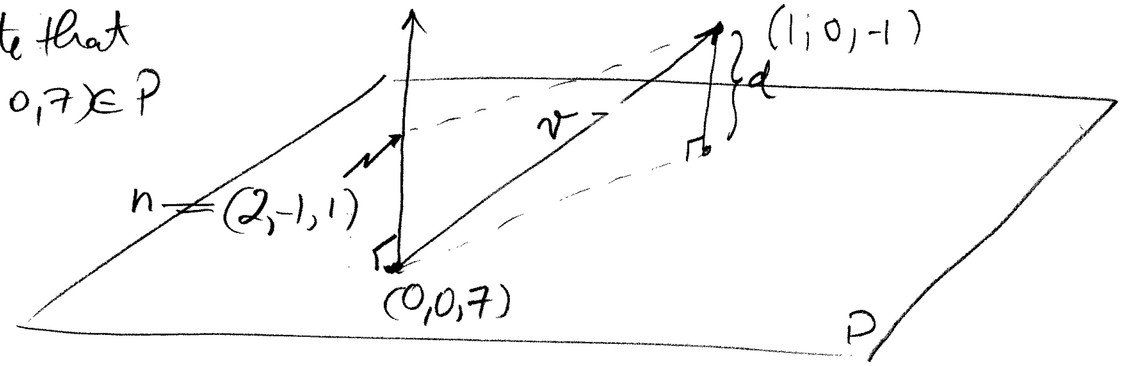
B. $2\sqrt{6}$

C. $\frac{\sqrt{6}}{2}$

D. $\frac{1}{6}$

E. $\frac{1}{3}$

F. $-2\sqrt{6}$



$$d = \| \operatorname{proj}_n v \| ; \; v = (1,0,-1) - (0,0,7) = (1,0,-8)$$

$$\operatorname{proj}_n v = \frac{(1,0,-8)\cdot(2,-1,1)}{\|(2,-1,1)\|^2}(2,-1,1) = \frac{-6}{6}\cdot(2,-1,1)$$

$$\therefore \; \| \operatorname{proj}_n v \| = \|(2,-1,1)\| = \sqrt{6}$$

2. Consider the following two lines given in scalar parametric form:

$$L_1 = \{(x, y, z) \mid x = -2s + 1, \; y = s + 2, \; \text{and } z = 4s + 1, \text{where } s \in \mathbf{R}\} \quad d_1 = (-2,1,4)$$

$$L_2 = \{(x, y, z) \mid x = t + 3, \; y = t + 1, \; \text{and } z = t - 3, \text{where } t \in \mathbf{R}\} \quad d_2 = (1,1,1)$$

Which **one** of the following statements is correct?

A. L_1 and L_2 intersect at (3, 1, −3).

B. L_1 and L_2 intersect at (−3, 1, −3).

C. L_1 and L_2 intersect at (3, −1, −3).

D. L_1 and L_2 are parallel. ×

E. L_1 and L_2 are perpendicular. ×

F. L_1 and L_2 are not coplanar. (i.e. do not intersect!)

We solve
$$\left.\begin{aligned} -2s+1 &= t+3 \quad (1)\\ s+2 &= t+1 \quad (2)\\ 4s+1 &= t-3 \quad (3)\end{aligned}\right\}$$
(1)−(2): $-3s - 1 = 2 \therefore s = -1$, so $t = 0$ (from (2)); These values of s and t satisfy (3), so L_1 & L_2 intersect at (3, 1, −3).

3. The intersection of three planes in $\mathbf{R}^3$ is <u>always</u>

A. Empty

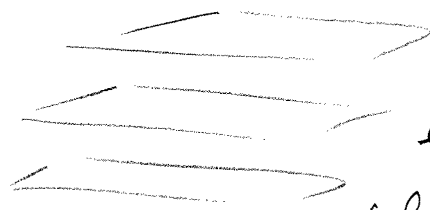
B. A line

C. A plane

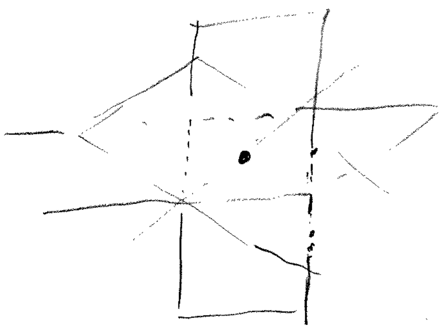
D. A point

E. A point, or a line, or a plane

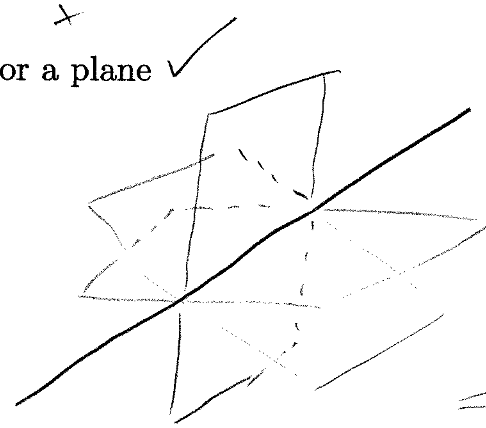
F. Empty, or a point, or a line, or a plane



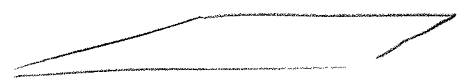
empty (planes are distinct & parallel)



Point



line



planes are identical

4. The equation $5x - y + 6z = -3$ is the equation of ... a plane

A. a ~~line~~ in $\mathbf{R}^3$ with direction vector $(5,\ -1,\ 6)$.

B. a plane passing through the points $(9,\ 0,\ -8)$, $(1\ ,1,\ 1)$ and $(0,\ 3,\ 0)$.

C. a plane with normal vector $(5,\ -1,\ 6)$ and passing through the point $(0,\ 3,\ 1)$.

D. a plane with normal vector $(5,\ -1,\ 6)$ and passing through the point $(9,\ 0,\ -8)$.

E. a ~~line~~ in $\mathbf{R}^3$ passing through the points $(0,\ 3,\ 0)$ and $(9,\ 0,\ -8)$

F. a plane, with normal vector $(0,\ \cancel{3},\ 0)$ and passing through the point $(5,\ -1,\ 6)$

The normal vector is $(5, -1, 6)$

$$5(9) - 0 + 6(-8) = -3$$

5. The volume of the parallelepiped with edges given by the vectors $u = (1, -1, 0)$, $v = (0, 1, 2)$ and $w = (2, 0, 1)$ is:

$|u \cdot v \times w|$

A. 1

B. 2

C. 3

D. 4

E. 5

F. 0

$$u \cdot v \times w = (1,-1,0) \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= (1,-1,0) \cdot (1, +4, -2)$$

$$= -3$$

$\therefore$ vol $= |-3| = 3$

6. Find the area of the triangle with vertices (2, 1, 0), (0, −1, 2) and (1, −2, 2).

A B C

A. $\sqrt{6}$

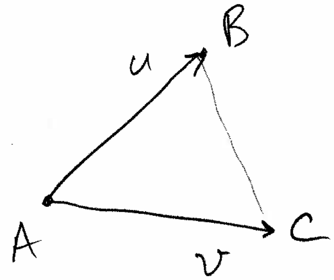
B. $2\sqrt{6}$

C. $\frac{\sqrt{6}}{2}$

D. $\frac{\sqrt{6}}{6}$

E. $\frac{\sqrt{6}}{3}$

F. $4\sqrt{6}$



area $= \frac{1}{2} \|u \times v\|$

But

$u = B - A = (-2,-2,2)$

$v = C - A = (-1,-3,2)$, so

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -2 & 2 \\ -1 & -3 & 2 \end{vmatrix} = (2, +2, 4). \quad \therefore \frac{1}{2}\|u \times v\| =$$

$$\frac{1}{2}\sqrt{2^2 + 2^2 + 4^2}$$

$$= \frac{1}{2}\underbrace{\sqrt{24}}_{\sqrt{4 \cdot 6}} = \frac{1}{2} \cdot 2 \cdot \sqrt{6} = \sqrt{6}$$

7. A direction vector for the line of intersection of the planes with equations $x - 2y = 1$ and $x + y - z = 0$ is:

A. (2, 1, 3)

B. (−2, 1, 3)

C. (−2, −1, 3)

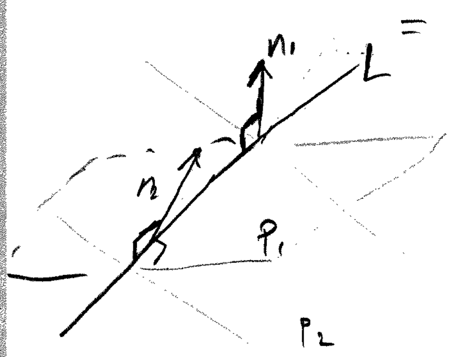
D. (2, −3, −1)

E. (−2, 3, −1)

F. (2, 3, −1)

Normals are $n_1 = (1,-2,0)$ & $n_2 = (1,1,-1)$

∴ a dirn vector for the line of intersection is $n_1 \times$



$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{vmatrix} = (2, +1, 3)$$

($L = P_1 \cap P_2$ lies in both planes so is $\perp$ to both their normals)

8. Parametric equations of the line containing (1, −1, 2) and which is parallel to the two planes with equations $x - y = 1$ and $x + y - 3z = 0$ are:

A. $x = 1 + 3t$, $y = -1 + 3t$, $z = 2 + 2t$, $t \in \mathbf{R}$

B. $x = 1 - 3t$, $y = -1 + 3t$, $z = 2 + 2t$, $t \in \mathbf{R}$ ×

C. $x = 1 - 3t$, $y = 1 + 3t$, $z = 2 + 2t$, $t \in \mathbf{R}$ ×

D. $x = 1$, $y = -1 + 3t$, $z = 2 + 2t$, $t \in \mathbf{R}$ ×

E. $x = 1 + 3t$, $y = 1$, $z = 2 + 2t$, $t \in \mathbf{R}$ ×

F. $x = 1 - 3t$, $y = -1$, $z = 2 + 2t$, $t \in \mathbf{R}$ ×

A line parallel to these lines will have a direction vector which is perpendicular to both their normals.

Here $n_1 = (1,-1,0)$
$n_2 = (1,1,-3)$,

so $n_1 \times n_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 1 & -3 \end{vmatrix} = (3, +3, 2)$; the only possible soln is A, which does pass through (1,−1,2)

9. Find a scalar equation for the plane

$$H = \{(1+s+2t, 2+t, 1+s) \mid s,t \in \mathbf{R}.\}$$

$$= \{(1,2,1) + s(1,0,1) + t(2,1,0) \mid s,t \in \mathbb{R}\}$$

A. $x - 2y - z = 4$?

B. $x + 5y - 2z = 9$ ✗

C. $-x + 2y + z = 4$?

D. $x - 2y + z = -3$ ✗

E. $-x - 2y + z = -4$ ✗

F. $3x + 2y - z = -2$ ✗

The plane will be ⊥ to both $(1,0,1)$ and $(2,1,0)$ and so will have normal $(1,0,1)\times(2,1,0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix}$

$= (-1, +2, 1)$

The eqn is then $-1(x-1) + 2(y-2) + 1.(z-1) = 0$

or $-x + 2y + z = 4$.

10. If $u = (1,\ 1,\ 1)$ and $v = (2,\ 1,\ 3)$ find the orthogonal projection of u on v, that is, $\text{proj}_v u$.

A. $\frac{3}{7}(2,\ 1,\ 3)$

B. $\frac{4}{7}(2,\ 1,\ 3)$

C. $\frac{3\sqrt{14}}{7}(2,\ 1,\ 3)$

D. $(1,\ 1,\ 1)$ ✗

E. $\frac{3}{7}(3,\ 3,\ 3)$ ✗

F. $2\sqrt{3}\ (1,\ 1,\ 1)$ ✗

$$\text{proj}_v u = \frac{u \cdot v}{\|v\|^2} v$$

$$= \frac{(1,1,1)\cdot(2,1,3)}{(4+1+9)}(2,1,3)$$

$$= \frac{6}{14}(2,1,3) = \frac{3}{7}(2,1,3)$$

11. Evaluate Im(z) if

$$z = \frac{1-3i}{1+i}. \qquad = \frac{(1-3i)\cdot(1-i)}{|1-i|^2}$$

$$= \frac{(1-3i)(1-i)}{1^2+(-1)^2}$$

$$= \frac{1 \mp 3 - 4i}{2}$$

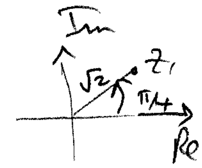
$$= \frac{\overset{-2}{\cancel{4}} - 4i}{2} = \overset{-1}{\cancel{2}} - 2i$$

A. 2
B. −2
C. −1
D. 1
E. 3
F. −3

12. Find the polar form of:

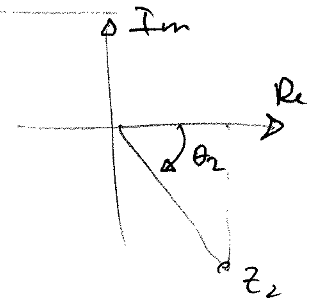
$$z = \frac{1+i}{1-\sqrt{3}i} = \frac{z_1}{z_2}; \qquad z_1 = |z_1| e^{i\theta_1}$$
$$z_2 = |z_2| e^{i\theta_2}$$
$$= \frac{|z_1|}{|z_2|} e^{i(\theta_1-\theta_2)}$$

A. $\frac{\sqrt{2}}{2}\Big(\cos(-\frac{7\pi}{12}) + i\sin(-\frac{7\pi}{12})\Big)$
B. $\frac{\sqrt{2}}{2}\Big(\cos(\frac{7\pi}{12}) + i\sin(\frac{7\pi}{12})\Big)$
C. $\sqrt{2}\Big(\cos(-\frac{\pi}{12}) + i\sin(-\frac{\pi}{12})\Big)$
D. $\sqrt{2}\Big(\cos(\frac{5\pi}{12}) + i\sin(\frac{5\pi}{12})\Big)$
E. $\sqrt{2}\Big(\cos(-\frac{5\pi}{12}) + i\sin(-\frac{5\pi}{12})\Big)$
F. $\frac{\sqrt{2}}{2}\Big(\cos(\frac{\pi}{12}) + i\sin(\frac{\pi}{12})\Big)$



$z_1 = 1+i, \quad |z_1| = \sqrt{1^2+1^2} = \sqrt{2}$

$\cos\theta_1 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \therefore \theta_1 = \pi/4$
$\sin\theta_1 = \frac{\sqrt{2}}{2}$



$z_2 = 1 - \sqrt{3}i$
$|z_2| = \sqrt{1^2 + (-\sqrt{3})^2} = 2$

$\cos\theta_2 = \frac{1}{2} \qquad \therefore \theta_2 = -\pi/3$
$\sin\theta_2 = -\frac{\sqrt{3}}{2}$

$\therefore \theta_1 - \theta_2 = \frac{\pi}{4} + \pi/3 = \frac{3+4}{12}\cdot\pi = \frac{7\pi}{12} \qquad \therefore z = \frac{\sqrt{2}}{2} e^{i\frac{7\pi}{12}}$