MAT 1330, Fall 2014 Assignment 6
Due Thursday November 20 by 9:00pm.
Late assignments will not be accepted; nor will unstapled assignments. Professors in the math department will not lend you a stapler; do not ask for one.

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Student Name ____________________________ Student Number ____________________________
By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ____________________________

QUESTION 1. (a) Use the mean value theorem (MVT) to show that there is a number $c \in [1, 2]$ where the function $f(x) = \frac{1}{x}$ has slope -1/2. Find the value of c.

We begin with the conclusion of the MVT:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

where $f'(x) = \frac{-2}{x^2}$. We want to show that there is a number c in $[0, 2]$ such that the slope of the function at that point is $-1/2$. In other words, we want to show that there is a number c such that the slope of the tangent line at that point is $-1/2$. We have

$$f'(c) = -1/2 \Leftrightarrow \frac{-2}{c^2} = \frac{-1}{2} \Leftrightarrow c^2 = 4.$$

Therefore, $c = 2$ corresponds to the point where the function $f(x) = \frac{1}{x}$ has a slope of $-1/2$ in the interval $[1, 2]$.

(b) Now consider the same function $f(x) = \frac{1}{x}$ on the interval $[a, b]$ with $0 < a < b < \infty$. What is the value of the derivative that the MVT guarantees exists? At which point $c \in [a, b]$ does it occur?

We again begin with the conclusion of the MVT

$$f'(c) = \frac{f(b) - f(a)}{b - a} \Leftrightarrow f'(c) = \frac{\frac{1}{b} - \frac{1}{a}}{b - a}.$$

Simplifying the above expression, we obtain the value of the derivative that the MVT guarantees exists which is

$$f'(c) = \frac{-1}{ab}.$$

The point c in $[a, c]$ where this value of the derivative occurs is obtained from the following equation

$$\frac{-2}{c^2} = \frac{-1}{ab} \Leftrightarrow c^2 = 2ab.$$

Since $a > 0$, the point c is given by $c = \sqrt{2ab}$.

(c) Now consider the same function $f(x) = \frac{1}{x}$ on the interval $[-1, 1.]$ Since $f(-1) = -1$ and $f(1) = 1$ there should be a point $c \in [-1, 1]$ such that $f'(c) = \frac{1-(-1)}{1-(-1)} = 1$, according to the MVT. Calculate f' and show that no such c can exist? What is wrong in the previous reasoning?

The derivative of f is given by $f'(x) = \frac{-2}{x^2}$. From the conclusion of the MVT, we obtain the equation

$$\frac{-2}{c^2} = 1 \Leftrightarrow c^2 = -2$$

which has no solution. Therefore, no such c can exist. Note that, in this example, the MVT does not apply since the function $f(x) = \frac{1}{x}$ is not continuous on the interval $[-1, 1]$.

Question 2. Consider a population that grows according to the logistic updY(hating function and is harvested at a linear rate $h \geq 0$. The number of individuals of the species satisfies the DTDS

$$x_{t+1} = x_t(4 - x_t) - hx_t.$$

(a) Determine all equilibria of the DTDS.

(b) Determine conditions on h such that all equilibria are biologically relevant.

(c) Determine conditions on h such that the positive equilibrium is stable.

(d) Determine conditions on h such that the positive equilibrium is unstable.

(e) Determine the value of h that maximizes the yield and state the resulting maximum yield.

(a) Equilibria of the DTDS satisfy

$$x^* = x^*(4 - x^*) - hx^* \Leftrightarrow 0 = x^*\big(4 - x^* - h - 1\big).$$

Thus, the equilibria of the DTDS are $x^* = 0$ and $x^* = 3 - h$.

(b) In order for all equilibria to be biologically relevant, we require $3 - h > 0$ or $h < 3$. Therefore, the conditions on h are $0 \leq h < 3$.

(c) The derivative of the updating function if $f'(x) = 4 - 2x - h$. Evaluating the derivative at the positive equilibrium, we find

$$f'(x^*) = 4 - 2(3 - h) - h = h - 2.$$

The equilibrium is stable when $\big|f'(x^*)\big| < 1$; i.e. $|h - 2| < 1$. The solution to this absolute value inequality is $1 < h < 3$. Therefore, the positive equilibrium is stable when $1 < h < 3$.

(d) Based on the results of part (c), we see that the equilibrium is unstable when $0 \leq h < 1$.

(e) The yield function is given by $Y(h) = hx^* = h(3 - h)$. The derivative of this function if $Y'(h) = 3 - 2h$ and the critical point is $h = 3/2$. Since the yield function is a concave down quadratic, we know that this critical point corresponds to a maximum. Thus, $h = 3/2$ maximizes the yield and the maximum yield is given by $Y(h) = 3/2(3 - 3/2) = 9/4$.

QUESTION 3. Consider the functions $f(x) = e^{x/3}$ and $g(x) = 2 - x^2/2$.
(a) Show that the functions intersect in the interval $[0, 2]$.

First, note that both f and g are continuous on $[0, 2]$. In fact, both functions are continuous for all x. The intersection of f and g satisfy

$$e^{x/3} = 2 - x^2/2 \Leftrightarrow 2 - x^2/2 - e^{x/3} = 0.$$

We then define the function $h(x) = 2 - x^2/2 - e^{x/3}$. Then, zeros of h correspond to intersection points of f and g. Since h is continuous, the IVT applies. Evaluating h at the end points of $[0, 2]$ we find

$$h(0) = 1 \text{ and } h(2) = -e^{2/3}.$$

Since h is above the x-axis at $x = 0$ and below the x-axis at $x = 2$, h has a zero in $[0, 2]$ and, therefore, the functions f and g intersect in $[0, 2]$.

(b) Use Newton's method to calculate the intersection point. Write down the general formula for Newton's method. Then start with $x_1 = 1$ and do three iterations.

Newton's method is given by the following DTDS

$$x_{t+1} = x_t - \frac{h(x_t)}{h'(x_t)}$$

where $h'(x) = -x - \frac{e^{x/3}}{3}$. Starting with $x = 1$ we find

$$\begin{aligned} x_2 &= x_1 - \frac{h(x_1)}{h'(x_1)} = 1 - \frac{.1044}{-1.4652} = 1.0713 \\ x_3 &= x_2 - \frac{h(x_2)}{h'(x_2)} = 1.0713 - \frac{-0.0030}{-1.5365} = 1.0693 \\ x_4 &= x_3 - \frac{h(x_3)}{h'(x_3)} = 1.0693 - \frac{.00007224}{-1.5345} = 1.0693 \end{aligned}$$

We can check the accuracy of our method by evaluating f and g at x_4. We find

$$f(x_4) = f(1.0693) = 1.4282 \text{ and } g(x_4) = g(1.0693) = 1.428.$$

Thus, we have found the intersection point of f and g.

QUESTION 4. The goal of this question is to show that the function $f(x) = x^3 + x^2 + 3x + 2$ for $x \in (-\infty, \infty)$ has exactly one zero. We split this question into a few subquestions.
(a) Use the intermediate value theorem to show that there exists (at least) one zero.

Since the function is a polynomial, it is continuous for all x and hence the IVT applies. Evaluating the function at the end points of the closed interval $[-4, 0]$ we find

$$f(-4) = -58 \text{ and } f(0) = 2.$$

Hence, f is below the x-axis at $x = -4$ and above the x-axis at 0. By the IVT, f has a root in $[0, 4]$.

(b) Use Rolle's theorem to show that if there are two (or more) zeros, then there is at least one critical point.
We assume that there exist two real numbers, a and b, such that $f(a) = f(b) = 0$; i.e. a and b are zeros of f. Since f is continuous and differentiable for all x, all conditions of Rolle's Theorem are satisfied. Thus, there exists a real number c such that $a < c < b$ where $f'(c) = 0$.

(c) Show that the function f does not have a critical point.

The first derivative of f is $f'(x) = 3x^2 + 2x + 3$. Critical points satisfy $f'(x) = 0$, i.e. $3x^2 + 2x + 3 = 0$. Since the discriminant of this quadratic equation is $\Delta = b^2 - 4ac = -20$, there are no real roots to this equation. Therefore, f does not have any critical points which contradicts the conclusion of Rolle's Theorem. This implies that f has only one real root.