MAT 1330, Fall 2014 Assignment 5

Due Thursday November 13 by 9:00pm.

Late assignments will not be accepted; nor will unstapled assignments. Professors in the math department will not lend you a stapler; do not ask for one.

Instructor (circle one): Frithjof Lutscher   Jason Levy   Farnoosh Babazadeh   Jeff Musgrave

DGD (circle one): 1   2   3   4

Student Name ____________________________ Student Number ____________________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ____________________________

QUESTION 1. Find the point on the curve $y = \sqrt{x}$ closest to the point (10,0). Hint: minimize the square of the distance from (10,0) to (x, y).

A point (x, y) on the curve will be of the form $(x, \sqrt{x})$. The distance from (10,0) to this point is a square root; write $g(x)$ for the square of the distance. The domain of $g(x)$ is all non-negative real numbers, $x \geq 0$. The formula for $g(x)$ is

$$g(x) = (x - 10)^2 + (\sqrt{x} - 0)^2 = x^2 - 19x + 100.$$

We want to find which x value in the domain of g minimizes the value of $g(x)$. This will be at a critical point of g or an endpoint of the domain. When $x = 0$ (the left endpoint of the domain), we have $g(x) = g(0) = 100$, and as $x \to \infty$ (the right "endpoint" of the domain) we have $g(x) \to \infty$.

How about critical points? Well, $g'(x) = 2x - 19$, which is always defined. It equals 0 exactly for $x = 19/2$. This is the only critical point. The value of g there is $g(19/2) = (19/2)^2 - 19(19/2) + 100 = 100 - \frac{361}{4} = 39/4$.

Comparing the values of g at the critical point and endpoints shows that the minimum distance occurs at $x = 19/2$. Now we go back and read the question: it asks not only for the x value, but also for the corresponding y value of the point on the curve.

Answer: $(19/2, \sqrt{19/2})$

QUESTION 2. Evaluate the following limits (do not use sequences of values; you may need to use L'Hopital's rule multiple times):

(*a*)

$$\lim_{x\to 0} \frac{e^x - 1 - x}{\cos(x) - 1}$$

Plugging in $x = 0$ to this limit gives the indeterminate form $0/0$, so we can apply L'Hopital's rule:

$$\lim_{x\to 0} \frac{e^x - 1 - x}{\cos(x) - 1} = \lim_{x\to 0} \frac{e^x - 0 - 1}{-\sin x}$$

Again, plugging in $x = 0$ gives $0/0$, so we can again apply L'Hopital's rule:

$$= \lim_{x\to 0} \frac{e^x}{-\cos x}$$

This time, plugging in $x = 0$ gives $1/(-1) = -1$, so we can't apply L'Hopital again, but we don't need to: continuity implies that this -1 is the answer

$$= -1$$

Limit equals | -1 |

(*b*)

$$\lim_{x\to\infty} (x+3)^{1/x}$$

Plugging in $x = \infty$ to this limit gives the indeterminate form ∞^0. To use L'Hopital's rule we need to change this limit to one of the appropriate type. Notice that

$$(x+3)^{1/x} = e^{\ln\left((x+3)^{1/x}\right)} = e^{\frac{\ln(x+3)}{x}},$$

and plugging $x = \infty$ into the fraction $\frac{\ln(x+3)}{x}$ gives the appropriate form ∞/∞. Therefore by L'Hopital's rule,

$$\lim_{x\to\infty} \frac{\ln(x+3)}{x} = \lim_{x\to\infty} \frac{1/(x+3)}{1} = \lim_{x\to\infty} \frac{1}{x+3} = 0,$$

and so by continuity of the exponential function,

$$\lim_{x\to\infty} (x+3)^{1/x} = \lim_{x\to\infty} e^{\frac{\ln(x+3)}{x}} = e^0 = 1.$$

Limit equals | 1 |

Question 3. In a movie theatre, the screen on the wall is 20 m high and its base is 10 m above eye level. Let θ denote the viewing angle of the screen, that is, the angle $\angle BET$ from the bottom (B) of the screen to the top (T), measured from the vertex of your eye (E). At what distance x from the screen should you position yourself to maximize θ? (from D. Kouba)

Let's draw a picture, labelling the points, the angles, and the distance x:

The unknown distance x from point E to point B is measured in metres. For convenience we introduce the additional angle α between EB and the horizontal.

The algebraic relations coming from trigonometry are that

$$\tan(\alpha) = \frac{10}{x}, \qquad \tan(\alpha + \theta) = \frac{30}{x};$$

solving for θ gives

$$\theta = \arctan \frac{30}{x} - \arctan \frac{10}{x}.$$

This is the function $\theta(x)$ of x that we want to maximize. The domain is the set of all $x > 0$ (since x represents a distance).

As $x \to \infty$, it's clear that $\theta(x)$ approaches 0. On the other hand, as $x \to 0^+$, both $\frac{30}{x}$ and $\frac{10}{x}$ approach ∞; since $\lim_{t\to\infty} \arctan(t) = \pi/2$, we see that

$$\lim_{x\to 0} \theta(x) = \lim_{x\to 0} \arctan \frac{30}{x} - \arctan 10x = \pi/2 - \pi/2 = 0.$$

So at both "endpoints" of the domain, the function θ approaches 0.

Since $\theta(x) > 0$ for any $x > 0$, the maximum of $\theta(x)$ will therefore occur at a critical point; let's determine them. By the chain rule,

$$\theta'(x) = \frac{1}{1 + \left(\frac{30}{x}\right)^2}(-30x^{-2}) - \frac{1}{1 + \left(\frac{10}{x}\right)^2}(-10x^{-2}) = \frac{-30}{x^2 + 30^2} + \frac{10}{x^2 + 10^2}.$$

This is always defined, and equals 0 when $\frac{30}{x^2+30^2} = \frac{10}{x^2+10^2}$, or equivalently $30(x^2 + 10^2) = 10(x^2+30^2)$, or $20x^2 = 6000$, $x^2 = 300$, $x = 10\sqrt{3}$ (remember that we do not consider negative values for x).

So there is only one critical point, $x = 10\sqrt{3}$, at which the angle $\theta(x)$ equals

$$\arctan \frac{30}{x} - \arctan \frac{10}{x} = \arctan \sqrt{3} - \arctan \frac{1}{\sqrt{3}} = \pi/3 - \pi/6 = \pi/6.$$

Comparing θ at the endpoints and at this critical point, we see that the largest θ is $\pi/6$, occuring at $x = 10\sqrt{3} \sim 17.32$.

Finally, we read the question again. It asks for x. So the answer is:

You should position yourself at a distance of $10\sqrt{3}$ from the screen to maximize θ.

QUESTION 4. (a) Find the linear approximation $L(x)$ to $f(x) = e^{2\sin x}$ around $x = \pi$.

The general for the linear approximation for $f(x)$ around $x = a$ is $L(x) = f(a)+f'(a)(x-a)$. Here we are taking $a = \pi$. The value of $f(a)$ is $e^{2\sin\pi} = e^0 = 1$. The formula for $f'(x)$ is, by the chain rule, $e^{2\sin x}(2\cos x)$; plugging in $x = a$ gives $f'(a) = e^0(2\cos\pi) = -2$. We plug these values into the formula for $L(x)$:

$L(x) =$ $1 - 2(x - \pi)$

(b) Use your answer in part (a) to estimate $e^{2\sin(3)}$.

The desired number is $f(3)$, which we can approximate by the value of $L(3) = 1-2(3-\pi) = 2\pi - 5 \sim 1.283$

Answer: $2\pi - 5$

(c) Find the cubic approximation $T_3(x)$ to $f(x) = e^{2x}$ at $x = 0$.

The general formula for the cubic approximation of $f(x)$ around $x = a$ is

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3,$$

where $3!$ means $3 \times 2 \times 1 = 6$. Here the formulas for the derivatives are: $f'(x) = 2e^{2x}$, $f''(x) = 4e^{2x}$, $f'''(x) = 8e^{2x}$, and we evaluate them at $a = 0$ to get: $f(a) = 1$, $f'(a) = 2$, $f''(a) = 4$, $f'''(a) = 8$. Plugging in to the formula for T_3 gives

$$T_3(x) = 1 + 2(x-0) + \frac{4}{2}(x-0)^2 + \frac{8}{6}(x-0)^3 = 1 + 2x + 2x^2 + \frac{4}{3}x^3.$$

$T_3(x) =$ $1 + 2x + 2x^2 + \frac{4}{3}x^3$