MA121 Mock Test 1

**Name:** Agassi Iu

**Time Allowed:** 80 minutes
**Total Value:** 55 marks
**Number of Pages:** 5

**Instructions:**

*Non-programmable, non-graphing calculators are permitted. No other aids allowed.*

*Answer in the spaces provided.*

*Show all your work. Insufficient justification will result in a loss of marks.*

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| *If you would like some personal feedback on your work by having it graded, then submit your completed mock test paper to Tina Balfour by* ***Wednesday Feb 4th at 4:00pm.*** |
| *(BA432; slide your paper under my office door if I don't happen to be in when you drop by.)* |

**** Please remember that mock tests are meant as a means of providing an extra set of practice questions and basis for a review class. Do not study for the midterm based solely on the topics covered by the mock test! Go back through notes/assignments/homework to ensure you have reviewed all concepts discussed in the course.

$\{c\}$ $\{\{c\}\}$

1. [*6 marks*] Let $A = \{a, b, c, e\}$, $B = \{3, c, \{c\}, \{c, d\}\}$, $C = \{1, 2, 3, a, c, d\}$ and $D = \{1, \{c\}\}$.

(a) How many elements does the set $B$ have?

4

(b) State a subset of $C$ which contains exactly three elements.

B ✓ $\{1,2,3\}$

(c) Determine: $(A \cup B) \cap (A \cup D)$

$(A\cup B)\cap(A\cup D) = A\cup(B\cap D)$

$= \{a,b,c,e\} \cup [\{3,c,\{c\},\{c,d\}\} \cap \{1,\{c\}\}]$

$= \{a,b,c,e\} \cup \{\{c\}\}$

$= \{a,b,c,\{c\},e\}$

(d) Are any of the sets disjoint? Explain your answer.

yes A and D are disjoint since $A \cap D = \emptyset$ ✗

2. [4 *marks*] If $A = \{x \in \mathbb{R} \mid x < 3\}$, $B = \{x \in \mathbb{R} \mid -2 \leq x < 8\}$ and $C = \{x \in \mathbb{N} \mid -2 \leq x < 8\}$, determine

(a) $A \cup B$

$A \cup B$

$= \{x \in \mathbb{R} \mid x < 3\} \cup \{x \in \mathbb{R} \mid -2 \leq x < 8\}$

$= \{x \in \mathbb{R} \mid x < 8\}$

(b) $A \cap C$

$A \cap C$

$= \{x \in \mathbb{R} \mid x < 3\} \cap \{x \in \mathbb{N} \mid -2 \leq x < 8\}$

$= \{x \in \mathbb{N} \mid -2 \leq x < 3\}$

3. [3 *marks*] For each of the following, state which, if any, of the statements are correct:

$A \in B \qquad B \in A \qquad A \subseteq B \qquad B \subseteq A \qquad A = B$

(a) $A = \{1, 2\}$ $\qquad B = \{1, 2, \{1, 2\}\}$ $\qquad$ $A \in B$ ; $A \subseteq B$

(b) $A = \{\varnothing, \{\varnothing, \varnothing\}\}$ $\qquad B = \{\{\varnothing\}, \varnothing, \{\varnothing\}\}$ $\qquad$ $A \subseteq B$; $B \subseteq A$ ; $A = B$

(c) $A = 3$ $\qquad B = \{2n + 1 : n \in \mathbb{N}\}$ $\qquad$ $A \in B$

4. [3 *marks*] Consider the statement: $(p \vee q) \rightarrow \sim (p \vee r)$

(a) Give the converse of the statement.

$\sim (p \vee r) \rightarrow (p \vee q)$

(b) Give the contrapositive of the original statement.

$(p \vee r) \rightarrow \sim (p \vee q)$

(c) Give the contrapositive of the converse of the original statement.

contrapositive of $(\sim (p \vee r) \rightarrow (p \vee q))$

5. [2 *marks*] Let $p, q, r$ be statements, and let $A$ be a statement in terms of $p, q, r$ that has the truth values in the table shown to the right:

| $p$ | $q$ | $r$ | $A$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ |
| $F$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $F$ |

State the disjunctive normal form for statement $A$.

$p\wedge q\wedge r,\ p\wedge\sim q\wedge r,\ \sim p\wedge q\wedge\sim r$

$(p\wedge q\wedge r)\vee(p\wedge\sim q\wedge r)\vee(\sim p\wedge q\wedge\sim r)$

6. [3 *marks*] Determine if the statement $p \vee (q \rightarrow (\sim r \rightarrow p))$ is a tautology.

| $p$ | $q$ | $r$ | $\sim r\to p$ | $q\to(\sim r\to p)$ | $p\vee(q\to(\sim r\to p))$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | T | F | T | T | T |
| T | F | T | T | T | T |
| T | F | F | T | T | T |
| F | T | T | T | T | T |
| F | T | F | F | F | F |
| F | F | T | T | T | T |
| F | F | F | F | T | T |

No it is not a tautology

7. [4 *marks*] Use the Laws of Propositional Logic to show that $p \rightarrow\sim (q\wedge \sim r)$ is logically equivalent to $\sim p\vee \sim q \vee r$.

| $q\wedge\sim r$ | $p\to\sim(q\wedge\sim r)$ | $\sim p\vee\sim q$ | $\sim p\vee\sim q\vee r$ |
|---|---|---|---|
| F | T | F | T |
| T | F | F | F |
| F | T | T | T |
| F | T | T | T |
| F | T | T | T |
| T | T | T | T |
| F | T | T | T |
| F | T | T | T |

$p\to\sim(q\wedge\sim r)$

$\Rightarrow p\to\sim q\vee\sim(\sim r)$

$\Rightarrow p\to(\sim q\vee r)$

$\Rightarrow \sim p\vee(\sim q\vee r)$

8. [6 *marks*]

(a) Consider the statement: $\forall x,\ [(x \leq 2) \vee (x^2 + 4x - 5 \neq 0)]$.

Negate the statement, and simplify.

$(x+5)(x-1)\neq 0$

$x\neq -5, 1$

$\{\forall x\ \ x<-5,\ -5<x<1,\ 1<x\leq 2\}$

$\exists x, [(x>2)\wedge(x^2+4x-5=0)]$

(b) Let $X$ be the set of all people, $T$ the set of all time, and let $F(x,t)$ stand for fooling a person $x$ at time $t$.
Express Abraham Lincoln's quote:

"You can fool some of the people all the time, and all of the people some of the time, but you cannot fool all of the people all of the time."

using logical quantifiers, in terms of $X$, $T$ and $F(x,t)$.

$(\exists x\forall T)\wedge(\forall x\exists T)$

9. [14 *marks*] Prove or disprove each of the following. State the method of proof used.

(a) For every two sets $A$ and $B$, ~~$(A \cup B) - B = A$~~. if $A \cup B = A$, then $B \subseteq A$

Indirect: assume $B \not\subseteq A$

$\Rightarrow \exists x, x \in B$ and $x \notin A$

$\Rightarrow x \in A \cup B$ and $x \notin A$

$\Rightarrow A \cup B \neq A$

$\therefore$ Stmt is True (by indirect proof).

(b) ~~If $A$ and $B$ are disjoint nonempty subsets, then $A - B^c = \emptyset$.~~

If $A \cup B = A \cup C$ for any non-empty sets $A, B, C$, then $B = C$

Let $A = \{a, b, c, d\}$
$B = \{b, c\}$
$C = \{c, d\}$

$A \cup B = \{a, b, c, d\}$
$A \cup C = \{a, b, c, d\}$

$A \cup B = A \cup C$

but $B \neq C$

$\therefore$ Stmt is false (by counterexample)

(c) Let $x \in \mathbb{Z}$. Then $3x + 1$ is even if and only if $5x - 2$ is odd.

Biconditional!

Direct

$p \to q$:
Assume: If $3x+1$ is even, $3x+1 = 2k$ for $k \in \mathbb{Z}$

$3x = 2k - 1$

$\Rightarrow 5x - 2 = 2x + 3x - 2$
$= 2x + 2k - 1 - 2$
$= 2x + 2k - 2 - 1$
$= 2(x + k - 1) - 1$ for $x + k - 1 \in \mathbb{Z}$

$\Rightarrow 5x - 2$ is odd

Don't write $x = \frac{2k-1}{3}$

↑
not integer

$q \to p$:
Assume $5x - 2$ is odd

$5x - 2 = 2k + 1$ for $k \in$

$5x = 2k + 3$

$\Rightarrow 3x + 1 = 5x - 2x + 1$
$= 2k + 3 - 2x + 1$
$= 2k - 2x + 4$
$= 2(k - x + 2)$ for $k - x$
$\therefore$ ~~$3k$~~ $+1$ is even

$\therefore$ It is true (by two direct proofs)

(d) Let $n \in \mathbb{Z}$. If $n^2 + n + 1 < 0$, then $n$ is an odd integer.

the statement is false, $\Rightarrow p$ is false $p \to q$ is true.

$n^2 + n + 1 = (n + \frac{1}{2})^2 - \frac{1}{4} + 1$

$= (n + \frac{1}{2})^2 + \frac{3}{4} > 0$ for all $n$

always false, the stmt is true (vacuous proof

10. [5 *marks*] Let $\overbrace{a \text{ be an irrational number}}^{P_1}$ and $\overbrace{r \text{ be a nonzero rational number.}}^{P_2}$ Prove that if $\underbrace{s \text{ is a real number}}_{P_3}$, then $\underbrace{\text{either } ar+s \text{ or } ar-s \text{ is irrational.}}_{q}$

Note: if $q$ is/isn't irrational, use contradiction

$\sim q$ = $ar+s$ and $ar-s$ are rational

Assume: $a$ is irrational
$r$ is non-zero rational
$s$ is real
$ar+s$ is rational
$ar-s$ is rational

$\Rightarrow r = \frac{b}{c}$, $b, c \in \mathbb{Z}$, $c \neq 0$, $b \neq 0$

$\Rightarrow ar+s = \frac{d}{e}$, $d, e \in \mathbb{Z}$, $e \neq 0$

$\Rightarrow ar-s = \frac{f}{g}$, $f, g \in \mathbb{Z}$, $g \neq 0$

$\Rightarrow (ar+s)+(ar-s) = 2ar$

$\frac{d}{e} + \frac{f}{g} = 2a\frac{b}{c}$

$\left(\frac{c}{2b}\right)\frac{dg+ef}{eg} = a$

$a = \frac{cdg+cef}{2beg}$ for $cdg+cef \in \mathbb{Z}$, $2beg \in \mathbb{Z}$, $b, e, g \neq 0$

$\Rightarrow$ $a$ is rational
contradicts $P_1$ (*)

$\therefore$ Stmt is true (by contradiction)

11. [6 *marks*] Prove by induction: For all natural numbers $n$, $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.

1) $n = 1$ true

L.S. $= \frac{1}{1^2} = 1$ $\quad 1 \leq 1$

R.S. $= 2 - \frac{1}{1} = 1$

$\therefore$ L.S. $\leq$ R.S. is true for $n = 1$

2i) $n = k$ for $k \in \mathbb{N}$ I.H.

Let. $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$ for some arbitrary $k \in \mathbb{N}$

$\frac{1}{k^2} > \frac{1}{k+1^2}$

induction step $\rightarrow$ ii) $n = k+1$ for $k+1 \in \mathbb{N}$

$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{(k+1)}$

- From I.H:

$1 + \frac{1}{4} + \frac{1}{9} + \ldots \frac{1}{k^2} \leq 2 - \frac{1}{k} \leq \left(2 - \frac{1}{k}\right) + \frac{1}{(k+1)} \leq 2 - \frac{1}{(k+1)}$

$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k} + \frac{1}{(k+1)^2} \leq \left(2 - \frac{1}{k}\right) + \frac{1}{(k+1)^2}$

$\leq 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right)$

$\leq 2 - \left(\frac{k^2+k+1}{k(k+1)^2}\right)$

$\leq 2 - \left(\frac{k(k+1)+1}{k(k+1)^2}\right)$

$\leq 2 - \left(\frac{1}{k+1} + \frac{1}{k(k+1)^2}\right)$

$\leq 2 - \frac{1}{k+1} - \frac{1}{k(k+1)^2}$

$2 - \frac{1}{k+1} - \frac{1}{k(k+1)^2} \leq 2 - \frac{1}{(k+1)}$ *

$\therefore$ The stmt is true (by induction) $\forall n \in \mathbb{N}$