Instructions:

Non-programmable, non-graphing calculators are permitted. No other aids allowed.

Answer in the spaces provided.

Show all your work. Insufficient justification will result in a loss of marks.

1. [6 *marks*] Let $A = \{a, b, c, e\}$, $B = \{3, c, \{c\}, \{c, d\}\}$, $C = \{1, 2, 3, a, c, d\}$ and $D = \{1, c\}$.

   (a) State the number of elements in the power set of B; i.e., determine $|\mathcal{P}(B)|$.

   \# of elements $= 2^{|B|} = 2^4 = 16$ ✓

   $|P(B)| = 2^0 \; 2^4$

   $|P(C)| = 2^6$

   (b) Determine: $A \triangle C$

   $A \triangle C = (A - C) \cup (C - A)$
   $= \{b, e\} \cup \{1, 2, 3, d\}$
   $= \{1, 2, 3, b, d, e\}$

   5/6

   (c) Determine $D \times A$.

   $D \times A = \{1, c\} \times \{a, b, c, e\}$
   $= \{(1,a), (1,b), (1,c), (1,e), (c,a), (c,b), (c,c), (c,e)\}$

2. [8 *marks*]

   (a) State the dual of the equation from set algebra: $(A \cup B \cup C)^c = (A \cup B)^c \cap (A \cup C)^c$

   $(A \cap B \cap C)^c = (A \cap B)^c \cup (A \cap C)^c$ ✓

   (b) Let $A = \{1, 2, 3, \ldots, 9, 10\}$. Give an example of two sets S and B such that $S \subseteq \mathcal{P}(A)$, $|S| = 4$, $B \in S$ and $|B| = 2$.

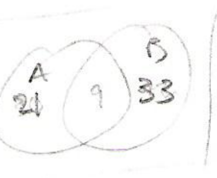
   4/6

   & S must be a set of sets'.

   $S = \{1, 2, 3, 4\}$

   $B = \{1, 2\}$ but $B \notin S$ here.

   (c) Given $|U| = 80$, $|A| = 30$, $|B| = 42$ and $|A \cap B| = 9$, determine $|A^c \cap B^c|$.

   $A^c = 50$
   $B^c = 38$

   $|A^c \cap B^c| = |(A \cup B)^c|$

   $|(A \cup B)|$
   $= 30 + 42 - 9 = 63$ ✓

   $|(A \cup B)^c| = 80 - 63 = 17$

3. [~~12~~ 14 *marks*] Let A, B and C be sets. Prove or disprove each of the following. State the method of proof used. ordered pairs !!

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

L.S. $x \in A$ and $(x \in B \cup C)$

iff $x \in A$ and $(x \in B$ or $x \in C)$

iff $(x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

iff $x \in (A \times B)$ or $x \in (A \times C)$

iff $(A \times B) \cup (A \times C)$

$(x,y) \in$

Let $(x,y) \in A \times (B \cup C)$

$\Leftrightarrow x \in A$ and $y \in (B \cup C)$

Let $(x,y) \in A \times (B \cup C)$

$x \in A$ and $y \in B \cup C$

$\Rightarrow x \in A$ and $(y \in B$ or $y \in C)$

$\Rightarrow (x \in A$ and $y \in B)$ or $(x \in A$ or $y \in C$

$\Rightarrow [(x,y) \in (A \times B)]$ or $[(x,y) \in (A \times C)]$

$\Rightarrow (x,y) \in (A \times B) \cup (A \times C)$

$\Rightarrow (A \times B) \cup (A \times C)$

$A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

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$\therefore$ is statement true or false ??

method ??

(b) $(A \cup B) - B = A$. ← This is actually false !

L.S. let $x \in A \cup B$ and $x \notin B$

iff $(x \in A$ or $x \in B)$ and $x \notin B$

iff $(x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin B)$

This works in one direction only! iff $x \in (A - B)$ or $x \in \emptyset$

You could use this to show $(A \cup B) - B \subseteq A$ but not equality.

iff $x \in A$

iff $x \in A$

$= R.S.$

$\therefore$ proved by direct proof ☑

$(A \cup B) - B = A$

if $A = B$

$A \cup B = A \cup A$

$= A$

$A \cup B - A = A - A$

$= \emptyset$

$\emptyset \neq A$

(c) If A and B are disjoint nonempty subsets, then $A - B^c = \emptyset$.

~~L.S.~~ Let $x \in A - B^c$ Then $x \in A$ and $(x \notin (U$ and $x \in B)$

iff $x \in A$ and $x \notin U$ and $x \in A$ and $x \in B$

iff $x \in A - U$ ← did you mean to erase this? and $x \in A \cap B$

iff $A - U = \emptyset$ ← ?? doesn't follow for all sets A and $A \cap B = \emptyset$ since they are distinct (disjoint).

$= \emptyset = R.S.$ ☑

$\therefore$ proved by direct proof

state this assumption at beginning.

1/4

$B^c = U - B$

$A - B^c = A - (U - B)$

$= A - U \oplus B$

?? there is no "addition" operation for sets

1/4

iff $x \in (A \times B)$ or $x \in (A \times C)$

iff $(A \times B) \cup (A \times C)$

$(x,y) \in$

$\Rightarrow x \in A$ and ($y \in B$ or $y \in C$)

$\Rightarrow$ ($x \in A$ and $y \in B$) or ($x \in A$ or $y \in C$)

$\Rightarrow [(x,y) \in (A \times B)]$ or $[(x,y) \in (A \times C)]$

$\Rightarrow (x,y) \in (A \times B) \cup (A \times C)$

$\Rightarrow (A \times B) \cup (A \times C)$

$A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

∴ is statement true or false??

method??

(b) $(A \cup B) - B = A$. ← This is actually false!

L.S. let $x \in A \cup B$ and $x \notin B$

iff ($x \in A$ or $x \in B$) and $x \notin B$

iff ($x \in A$ and $x \notin B$) or ($x \in B$ and $x \notin B$)

This works in one direction only! You could use this to show $(A \cup B) - B \subseteq A$ but not equality.

iff $x \in (A - B)$ or $x \in \emptyset$

iff $x \in A$

iff $x \in A$

$= R.S.$

∴ proved by direct proof ☑

$(A \cup B) - B = A$

if $A = B$

$A \cup B = A \cup A$

$= A$

$A \cup B - A = A - A$

$= \emptyset$

$\emptyset \neq A$

(c) If A and B are disjoint nonempty subsets, then $A - B^c = \emptyset$.

~~L.S.~~ Let $x \in A - B^c$. Then $x \in A$ and $x \notin$ (U and $x \in B$)

iff $x \in A$ and $x \notin U$ and $x \in A$ and $x \in B$

iff $x \in A - U^c$ ← did you mean to erase this? and $x \in A \cap B$

iff $A - U = \emptyset$?? ← doesn't follow. do all sets A and $A \cap B = \emptyset$ since they are distinct (disjoint).

state this assumption at beginning.

$= \emptyset = R.S.$ ☑

∴ proved by direct proof

1/4

$B^c = U - B$

$A - B^c = A - (U - B)$

$= A - U \oplus B$?? there is no "addition" operation for sets

~~Let $x \in A - B^c$~~

~~Then $x \in A$ and $x \notin (B^c)$~~

Contrapositive

Assume $A - B^c \neq 0$

$\Rightarrow \exists x \in A - B^c$

$\Rightarrow x \in A$ and $x \notin B^c$

$\Rightarrow x \in A$ and B

$\Rightarrow A$ and B are not disjoint

True

(d) If $A \subseteq B$, then $B^c \subseteq A^c$.

Assume $A \subseteq B$

~~$x \notin B$~~

Let $x \in B^c$

$\Rightarrow x \notin B$

$\Rightarrow x \notin A$

$\Rightarrow x \in A^c$

∴ $B^c \subseteq A^c$

6. [4 *marks*] Suppose $n \in \mathbb{Z}$, $n \geq \cancel{1}\,2$. Prove that $\binom{n+1}{2} - \binom{n}{2} = n$.

L.S. $\binom{n+1}{2} - \binom{n}{2} = \left[\binom{n+1-1}{2-1} + \binom{n+1-1}{2}\right] - \binom{n}{2}$

$= \left[\binom{n}{1} + \binom{n}{2} - \binom{n}{2}\right]$

$= \binom{n}{1}$ ✓

4/4

$= n$ for all n ////

7. [10 *marks*] Use the Binomial Theorem for each of the following.

(a) Find the term containing x^{12} in the expansion of: $\left(2x^6 - \frac{5}{x^3}\right)^{11}$.

$\left(2x^6 - \frac{5}{x^3}\right)^{11} = \binom{11}{0}(2x^6)^{11} + \binom{11}{1}(2x^6)^{10}\left(\frac{-5}{x^3}\right) + \binom{11}{2}(2x^6)^9\left(\frac{-5}{x^3}\right)^2 + \binom{11}{3}(2x^6)^8\left(\frac{-5}{x^3}\right)^3 +$

$+ \binom{11}{4}(2x^6)^7\left(\frac{-5}{x^3}\right)^4 + \binom{11}{5}(2x^6)^6\left(\frac{-5}{x^3}\right)^5 + \binom{11}{6}(2x^6)^5\left(\frac{-5}{x^3}\right)^6 + \binom{11}{7}(2x^6)^4\left(\frac{-5}{x^3}\right)^7 + \ldots$

3/6

$5^3 = 125$, 625, 3125, 15625

$\binom{11}{6}(2)^5(x^6)^5\left(\frac{-5}{x^3}\right)^6$

$\boxed{= \binom{11}{6}(32)(15625)\frac{x^{30}}{x^{12}}}$ ✓

→ not very efficient method!
→ need to learn how to use the general term to figure out "k".

$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$

$= 231\,000\,000\, x^{18}$

$\sum_{k=0}^{11}\binom{11}{k}(2x^6)^{11-k}\left(\frac{-5}{x^3}\right)^k$

$= cx^{12} = \binom{11}{k}2^{11-k}x^{66-6k}(-5)^k(x^{-3k})$

$12 = 66 - 9k$

$54 = 9k$

$k = 6$

(b) Determine the value of: $\sum_{k=1}^{2011} (-1)^k \binom{2011}{k} 5^k$.

$\binom{2011}{\ }\ldots k$

8. [12 *marks*]

(a) Divide the complex numbers $\dfrac{i+4}{5-\sqrt{3}i}$. Express your final answer in standard form..

$$\frac{i+4}{5-\sqrt{3}i} \times \frac{5+\sqrt{3}i}{5+\sqrt{3}i} = \frac{(i+4)(5+\sqrt{3}i)}{25-3i^2} = \frac{5i+4\sqrt{3}i+20+\sqrt{3}i^2}{28}$$

3/3

$$= \frac{(20-\sqrt{3})+5i+4\sqrt{3}i}{28}$$

$$= \frac{(20-\sqrt{3})}{28} + \frac{(5+4\sqrt{3})}{28}i$$

(b) Find all complex numbers z such that $|z| = 5$ and $z + \overline{z} = 6$.

$z = a+bi$

$z+\bar{z} = (a+bi)+(a-bi)$

$6 = 2a$

$a = 3$

1/3

$|z| = 5$

$|3+bi| = 5$

$\sqrt{(3)^2+(b^2)} = 5$

$\sqrt{3^2+b^2} = 25$

$b = 4, -4$

$\therefore 3+4i \quad 3-4i$

Prove:

(c) If z is a complex number such that $(\overline{z})^2 = z^2$, then z is pure real or z is pure imaginery.

let $z = a+bi$ → and let $(\bar{z})^2 = z^2$. ← need to indicate we are assuming this property holds for the direct proof

$\bar{z} = a-bi$

$(\bar{z})^2 = a^2-2abi+b^2i^2$

$= (a^2-b^2)-(2ab)i$

$(z)^2 = a^2+(2ab)i+b^2i^2$

$= (a^2-b^2)+(2ab)i$

2/3

$a^2-b^2-2abi = a^2-b^2+2abi$

$4abi = 0$

$a = 0$ and/or $b = 0$

z ~~can~~ must be pure real, pure imaginary or $\emptyset$.?

the empty set

if $z=0$, then this is a purely real complex number, not a separate case.

(d) Prove that if $z_1, z_2 \in \mathbb{C}$, then $|z_1+z_2|^2 + |z_1-z_2|^2 = 2|z_1|^2 + 2|z_2|^2$.

let $z_1 = a+bi$

$z_2 = c+di$

L.S. $|z_1+z_2|^2 + |z_1-z_2|^2$

good start!

3/3

$$\frac{(20-\sqrt{3}) + 5i + 4\sqrt{3}\,i}{28}$$

✓

$$= \frac{(20-\sqrt{3})}{28} + \frac{(5+4\sqrt{3})}{28}\,i$$

(b) Find all complex numbers z such that $|z| = 5$ and $z + \overline{z} = 6$.

1/3

$z = a+bi$

$z + \overline{z} = (a+bi) + (a-bi)$

$6 = 2a$ ✓

$a = 3$

$|z| = 5$

$|3+bi| = 5$

$\sqrt{(3)^2 + (b^2)} = 5$

$\sqrt{3^2 + b^2} = 25$

$b = 4, -4$

$\therefore$ $3+4i$ $\quad$ $3-4i$

Prove:

(c) If z is a complex number such that $(\overline{z})^2 = z^2$, then z is pure real or z is pure imaginery.

2/3

let $z = a+bi$ $\rightarrow$ and let $(\overline{z})^2 = z^2$. $\leftarrow$ need to indicate we are assuming this property holds for the direct proof.

$\overline{z} = a-bi$

$(\overline{z})^2 = a^2 - 2abi + b^2i^2$

$= (a^2-b^2) - (2ab)i$

$(z)^2 = a^2 + (2ab)i + b^2i^2$

$= (a^2-b^2) + (2ab)i$

$a^2-b^2-2abi = a^2-b^2+2abi$

$4abi = 0$

$a = 0$ and/or $b = 0$ ✓

z ~~can~~ must be pure real, pure imaginary or $\emptyset$?

the empty set

if $z=0$, then this is a purely real complex number, not a separate case.

(d) Prove that if $z_1, z_2 \in \mathbb{C}$, then $|z_1+z_2|^2 + |z_1-z_2|^2 = 2|z_1|^2 + 2|z_2|^2$.

1/3

let $z_1 = a+bi$

$z_2 = c+di$

L.S. $|z_1+z_2|^2 + |z_1-z_2|^2$ ✓ good start!

$= |(a+bi)+(c+di)|^2 + |(a+bi)-(c+di)|^2$

$= |(a+c)+(b+d)i|^2 + |(a-c)+(b-d)i|^2$

$= (a+c)^2 + (b+d)^2 + (a-c)^2 + (b-d)^2$

$= a^2+2ac+c^2+b^2+2bd+d^2+a^2-2ac+c^2+b^2-2bd+d^2$

$= 2a^2+2b^2+2c^2+2d^2$

$= 2(a^2+c^2) + 2(b^2+d^2)$

$= 2|z_1|^2 + 2|z_2|^2$