MA121 Mock Final Exam

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**** Please remember that mock tests are meant as a means of providing an extra set of practice questions and basis for a review class. Do not study for the midterm based solely on the topics covered by the mock test! Go back through notes/assignments/homework to ensure you have reviewed all concepts discussed in the course.

Contrapositive
$\sim q \to \sim p$
Contradiction
$\sim q \to p$

Time Allowed: 150 minutes
Total Value: 100 marks
Number of Pages: 8

Instructions:

Cheat Sheet: One 8.5" × 11" page of study notes (both sides) is allowed as a reference while completing the mock test. Please note, that the cheat sheet is permitted for the mock test only!!

No other aids allowed [no calculators].

*Check that your test paper has no missing, blank, or illegible pages. Note that test questions appear on **both** sides of the paper.*

Answer in the spaces provided.

Show all your work. Insufficient justification will result in a loss of marks.

1. [7 *marks*] Let p, q and r be statements.

   (a) Show that $q \wedge (p \to (\sim r \to p))$ is logically equivalent to q.

   (b) Find the disjunctive normal form for the contrapositive of the converse of: $(p \wedge q) \to r$.

2/4

converse = $r \to (p \wedge q)$ ✓

contra of conv. = $\sim(p \wedge q) \to \sim r$

2. [8 *marks*] Use induction on $n \in \mathbb{N}$ to prove that $1 + 5 + 9 + \cdots + (4n - 3) = 2n^2 - n$ for all $n \geq 1$.

1) Base Case $n=1$

L.S. $(4(1)-3) = 1$ ✓ L.S $=$ R.S.

R.S. $2(1)^2 - 1 = 1$ $\therefore$ true for $n=1$

2) I.H. Assume $n=k$ true for $k \in \mathbb{N}$ ✓

$1+5+9+\ldots+(4k-3) = 2k^2 - k$

3) I.S. Show $n=k+1$ true for $k+1 \in \mathbb{N}$

$1+5+9+\ldots+(4(k+1)-3) = 2(k+1)^2 - (k+1)$

8/8

L.S. $1+5+9+\ldots+(4k-3)+(4k+4-3)$

$= \underbrace{1+5+9+\ldots+(4k-3)}_{IH.} + (4k+4-3)$

$= 2k^2 - k + 4k + 1$

$= 2k^2 + 4k + 2 - k - 1$ ✓

$= 2(k^2 + 2k + 1) - (k+1)$

$= 2(k+1)^2 - (k+1)$

$=$ R.S. $\therefore$ True

$\therefore$ The stmt is true for all $n \in \mathbb{N}$ proved by induction ✓

3. [20 *marks*] Prove or disprove each of the following statements. State the method of proof used.

(a) Let A and B be sets. Then $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

$P - Q \Leftrightarrow P \cap Q^c$

$(A \cup B) - (A \cap B) =$

$(A \cup B) \cap (A \cap B)^c$

$= (A \cup B) \cap (A^c \cup B^c)$

Let ~~$x \in A$ and B~~ $x \in [(A \cup B) - (A \cap B)]$

$\Leftrightarrow x \in (A \cup B)$ and $x \notin (A \cap B)$

OR $x \notin$ ... wise, ... doesn't

2) I.H. Assume n=k true for, k∈N ✓

$1+5+9+\ldots+(4k-3)=2k^2-k$

3) I.S. Show n=k+1 true for, k+1∈N

$1+5+9+\ldots+(4(k+1)-3)=2(k+1)^2-(k+1)$

8/8

L.S. $1+5+9+\ldots+(4k-3)+(4k+4-3)$

$=\underbrace{1+5+9+\ldots+(4k-3)}_{IH.}+(4k+4-3)$

$=2k^2-k+4k+1$

$=2k^2+4k+2-k-1$ ✓

$=2(k^2+2k+1)-(k+1)$

$=2(k+1)^2-(k+1)$

$=R.S.$ ∴ True ☒

∴ The stmt is true for all n∈N proved by induction ✓

3. [20 *marks*] Prove or disprove each of the following statements. State the method of proof used.

(a) Let A and B be sets. Then $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

$P-Q \Leftrightarrow P \cap Q^c$

$(A\cup B)-(A\cap B)=$

$(A\cup B)\cap(A\cap B)^c$

$=(A\cup B)\cap(A^c\cup B^c)$

Let ~~x∈A and B~~

$x\in[(A\cup B)-(A\cap B)]$

$\Leftrightarrow x\in(A\cup B)$ and $x\notin(A\cap B)$

$\Leftrightarrow (x\in A$ or $x\in B)$ and $(x\notin A$ ~~and~~ OR $x\notin B)$ otherwise, this doesn't follow.

$\Leftrightarrow (x\in A$ and $x\notin B)$ or $(x\in B$ and $x\notin A)$

2/3

$\Leftrightarrow (x\in A-B)$ or $(x\in B-A)$

$\Leftrightarrow (A-B)\cup(B-A)$

$\Leftrightarrow R.S.$ ☒

∴ proved by direct proof ✓

"trivial" if q is log. T.

(b) Let x be an even integer. If $x^2 - x + 2 < 0$, then I will pass my MA121 exam.

If x is an even integer, $x=2n$ for $n\in\mathbb{Z}$

1/2

Then: $(2n)^2-2n+2 < 0$

$=4n^2-2n+2 < 0$ show this is always >0

$2(2n^2-n+1)$

$2n^2-n+1<0$

$(2n+1)(n-1)<0$

if $x=4$, $=2(2)$

$4^2-4+2=14>0$

disproved by counterexample.

haven't shown this yet! only showed it was false for one x-value.

The first stmt is logically false, hence

#3. Continued: Prove or disprove each of the following statements. State the method of proof used.

(c) If x and y are positive integers such that $x+y$ is odd, then one of x and y is even and the other is odd. Assume "q" is false

Case 1, x and y are both even

$\Rightarrow x=2m \quad m\in\mathbb{Z}$

$y=2n \quad n\in\mathbb{Z}$ ✓

$\Rightarrow x+y = 2m+2n = 2(m+n)$ for $m+n\in\mathbb{Z}$ $\therefore$ even

$x+y$ will be even

Case 2, x and y are both odd

5/6

$\Rightarrow x=2m+1 \quad m\in\mathbb{Z}$

$y=2n+1 \quad n\in\mathbb{Z}$ ✓

$\Rightarrow x+y = 2m+1+2n+1 = 2m+2n+2 = 2(m+n+1)$ for $m+n+1\in\mathbb{Z}$ $\therefore$ even

Case 3, x is even and y is odd

Not req'd, as Case 1 + 2 together show $\sim q \to \sim p$

so st't is true by contrapositive.

and similarly if x odd and y even

$x=2m \quad m\in\mathbb{Z}$

$y=2n+1 \quad n\in\mathbb{Z}$

$x+y = 2m+2n+1 = 2(m+n)+1$ for $m+n\in\mathbb{Z}$ $\therefore$ Odd

True, proved by constructive proof

(d) $\forall y\in\mathbb{Z}\ \exists x\in\mathbb{Z}\ [y>8\to x^2=y]$.

all $y\in\mathbb{Z}$

some $x\in\mathbb{Z}$

Let $y=13$, satisfy $y\in\mathbb{Z}$ and $y=13>8$

$x\in\mathbb{Z}$

2/2

$x^2=y=13 \quad x=\pm\sqrt{13}\notin\mathbb{Z}$

$\nexists x\in\mathbb{Z}$ such that $x^2=13$

$\therefore x$ is not an integer and there is no integer x such that $x^2=13$

disproved by counter example.

(e) Let $n\in\mathbb{N}$. Then $\overbrace{3\mid(2n^2+1)}^{p}$ if and only if $\overbrace{3\nmid n}^{q}$.

Let $3\mid n$, prove $3\nmid(2n^2+1)$

if $3\mid n$, then $n=3k$ for $k\in\mathbb{N}$

$\Leftrightarrow 3\mid(2n^2+1)$?? but trying to show something about the divisibility of $2n^2+1$

$= 3\mid(2(3k)^2+1)$ Then $2n^2+1=2(3k)^2+1$

$= 3\mid(2(9k^2)+1)$ $= \vdots$

2/7

$= 3\mid(18k^2+1)$ $= 3(6k^2)+1$

$\Rightarrow 3\nmid(2n^2+1)$

right idea here, just have to work a little on proper form for the proof.

Case 2, x and y are both odd

5/6

$\Rightarrow x = 2m+1 \quad m \in \mathbb{Z}$

$y = 2n+1 \quad n \in \mathbb{Z}$ ✓

$\Rightarrow x+y = 2m+1 + 2n+1 = 2m+2n+2 = 2(m+n+1)$ for $m+n+1 \in \mathbb{Z}$ $\therefore$ even

Case 3, x is even and y is odd

Not req'd, as Case 1 + 2 together show $\sim q \to \sim p$

and similarly if x odd and y even

$x = 2m \quad m \in \mathbb{Z}$

$y = 2n+1 \quad n \in \mathbb{Z}$

so st't is true by contrapositive.

$x+y = 2m + 2n+1 = 2(m+n)+1$ for $m+n \in \mathbb{Z}$ $\therefore$ Odd

True, proved by constructive proof

(d) $\forall y \in \mathbb{Z}\ \exists x \in \mathbb{Z}\ [y > 8 \to x^2 = y]$.

all $y \in \mathbb{Z}$

some $x \in \mathbb{Z}$

Let $y = 13$, satisfy $y \in \mathbb{Z}$ and $y = 13 > 8$

$x \in \mathbb{Z}$

2/2

$x^2 = y = 13 \quad x = \pm\sqrt{13} \notin \mathbb{Z}$

$\nexists x \in \mathbb{Z}$ such that $x^2 = 13$

$\therefore x$ is not an integer and there is no integer x such that $x^2 = 13$

disproved by counterexample.

(e) Let $n \in \mathbb{N}$. Then $\overbrace{3 \mid (2n^2+1)}^{p}$ if and only if $\overbrace{3 \nmid n}^{q}$. $\leftrightarrow$

Let $3 \mid n$, prove $3 \nmid (2n^2+1)$

if $3 \mid n$, then $n = 3k$ for $k \in \mathbb{N}$

$\Leftrightarrow 3 \mid (2n^2+1)$?? but trying to show something about the divisibility of $2n^2+1$

$= 3 \mid (2(3k)^2+1)$ Then $2n^2+1 = 2(3k)^2+1$

$=$ ⋮

2/7

$= 3 \mid (2(9k^2)+1)$

right idea here, just have to work a little on proper form for the proof.

$= 3(6k^2)+1$

$= 3 \mid (18k^2+1)$

$\Rightarrow 3 \nmid (2n^2+1)$

$3 \mid 18k^2$ but $3 \nmid (18k^2+1)$ for $k \in \mathbb{N}$

$\therefore 3 \nmid (2n^2+1)$

$\therefore$ Proved by Contrapositive

You have shown $\sim q \to \sim p$

so we know $p \to q$ is true.

But you still need to show $q \to p$ is also true

Contrapositive

① $\sim q \to \sim p$ Assume $3 \mid n$

$n = 3k, k \in \mathbb{N}$

$n^2 = 9k^2$

$2n^2 = 18k^2$

$2n^2+1 = 18k^2+1$

$= 3(6k^2)+1$

$3 \nmid (3(6k^2)+1)$ for $6k \in \mathbb{N}$

$3 \nmid (2n^2+1)$

② Direct Proof by cases $q \to p$

$n = 3k+1, k \in \mathbb{N}$

4. [18 *marks*] From lectures, we know the following are true:

   1. Two integers a and b are relatively prime, if and only if there are integers k and l such that $ka + lb = 1$.
   2. If $m, n, p \in \mathbb{Z}$ such that $p|mn$ with p and m relatively prime, then $p|n$.
   3. If $m, n, p \in \mathbb{Z}$ where p is prime and $p|mn$, then either $p|m$ or $p|n$.

Using any of these results where necessary, prove each of the following.

(a) If p is prime, then $\sqrt{p}$ is irrational.

If p is prime

$\gcd(p, m) = 1$ for $\forall m \in \mathbb{Z}$ where $p \nmid m$ ✓ true

$\forall n,\ n \nmid p$, factors of $p = 1, p$

→ use a proof by contradiction

Assume $\sqrt{p}$ is rational and p is prime $(\neg q \to p)$

$\sqrt{p} = \frac{a}{b}$, $a, b \in \mathbb{Z}, b \neq 0, \gcd(a,b) = 1$

$p = \frac{a^2}{b^2}$

$b^2 p = a^2$

$\Rightarrow p | a^2 \ (p | a \cdot a)$

$\Rightarrow p | a$ (by (3))

$\Rightarrow a = p \cdot q \in \mathbb{Z}$

$a^2 = p^2 \cdot q^2$

$\therefore pb^2 = p^2 q^2$

$\to b^2 = pq^2$

$p | b^2 \ (p | b \cdot b)$

$\Rightarrow p | b$ (by (3))

Then if $p|a$ and $p|b$, $\gcd(a,b) \geq p > 1$

$(\Rightarrow \Leftarrow)$

$\therefore$ Stmt is true proof by contradiction

(b) Every two consecutive integers are relatively prime.

Let n and $n+1$ be 2 consecutive integers $(n \in \mathbb{Z})$

$(n+1) = 1 \cdot n + 1$

$n = n(1) + 0$

$1 = (n+1) - (n)$

$\gcd(n, n+1) = 1$

$\therefore$ n and $(n+1)$ are relatively prime.

OR using (1):

$k(n+1) + l(n) = 1$ if $k = 1$ $l = -1$

$\therefore$ as such k and l exists, $(n+1)$ and n are relatively prime.

Using any of these results where necessary, prove each of the following.

(a) If p is prime, then $\sqrt{p}$ is irrational.

If p is prime

$\gcd(p, m) = 1$ for $\forall m \in \mathbb{Z}$ where $p \nmid m$ ✓ true

$\forall n,\ n \nmid p$, factors of $p = 1; p$

→ use a proof by contradiction.

Assume $\sqrt{p}$ is rational and p is prime $(\neg q \to p)$

$\sqrt{p} = \frac{a}{b}$, $a, b \in \mathbb{Z}, b \neq 0$, $\gcd(a,b) = 1$

$p = \frac{a^2}{b^2}$

$b^2 p = a^2$

$\Rightarrow p \mid a^2$ $(p \mid a \cdot a)$

$\Rightarrow p \mid a$ (by (3))

$\Rightarrow a = p \cdot q \in \mathbb{Z}$

$a^2 = p^2 \cdot q^2$

$\therefore pb^2 = p^2 q^2$

$b^2 = pq^2$

$p \mid b^2$ $(p \mid b \cdot b)$

$= p \mid b$ (by (3))

Then if $p \mid a$ and $p \mid b$, $\gcd(a,b) \geq p > 1$

$(\Rightarrow \Leftarrow)$

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<u>OR</u> using (1):

$k(n+1) + l(n) = 1$ if $k = 1$ $l = -1$

$\therefore$ as such k and l exists, $(n+1)$ and n are relatively prime.

#4. Continued...

(c) If $n, k \in \mathbb{N}$ with $k \leq n$ and $\gcd(k, n) = 1$, then $n \left| \binom{n}{k} \right.$.

[Note: You can assume $\binom{n}{k} \in \mathbb{N}$ for all $n, k \in \mathbb{Z}$ such that $n \geq k \geq 0$.]

$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

$n \mid \binom{n}{k}$

$= n \mid \frac{n!}{(n-k)!\,k!}$

This is what we are trying to prove!

$\gcd(k,n)=1$, $\frac{n!}{(n-k)!\,k!} \in \mathbb{N} = mn$ for $m \in \mathbb{N}$

we don't know this yet!

~~proved by direct proof~~

Assume $n, k \in \mathbb{N}$, $\gcd(n,k)=1$

$\binom{n}{k} = n \cdot (\mathbb{Z})$

$= \frac{n!}{(n-k)!\,k!}$

$= n \cdot \frac{(n-1)!}{k!\,(n-k)!}$

$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

$k\binom{n}{k} = n\binom{n-1}{k-1}$

$k\binom{n}{k} = n\binom{n-1}{k-1}$ where $\in \mathbb{Z}$

$\Rightarrow n \mid k\binom{n}{k}$ since $\gcd(n,k)$, $n \mid \binom{n}{k}$ $\boxtimes$ proof by direct

5. [*8 marks*]

(a) Determine the prime factorization of (i) 180 and (ii) 252.

i) $180 = 2^2 \cdot 3^2 \cdot 5$

✓ OK, but show some work to justify answer!

ex $180 = 2 \times 90$
$= 2 \times 9 \times 10$
$= 2^2 \times 3^2 \times 5$

ii) $252 = 2^2 \cdot 3^2 \cdot 7$

$252 = 2 \times 126$
$= 2 \times 2 \times 63$
$= 2 \times 2 \times 9 \times 7$
$= 2 \times 2 \times 3 \times 3 \times 7$

(b) Use the results of part(a) to determine the greatest common divisor (180, 252) and the least common mulitple [180, 252].

8/8

$\gcd(180, 252) = 2^2 \cdot 3^2$
$= 4 \cdot 9$ ✓
$= 36$

$[180, 252] = \frac{180 \cdot 252}{\gcd(180, 252)}$ or $2^2 \cdot 3^2 \cdot 5 \cdot 7 = 1260$ ✓

(c) Apply the Euclidean Algorithm to determine integers k and l such that $(180, 252) = 180k + 252l$.

$252 = 1 \cdot 180 + 72$ $\qquad 72 = 252 - 180$

$n \mid \binom{n}{k}$ This is what we are trying to prove!

$= n \mid \frac{n!}{(n-k)!k!}$

$gcd(k,n)=1$, $\frac{n!}{(n-k)!k!} \in \mathbb{N} = mn$ for $m \in \mathbb{N}$

we don't know this yet!

proved by direct proof

Assume $n, k \in \mathbb{N}$, $gcd(n,k)=1$

$\binom{n}{k} = n \cdot (\mathbb{Z})$

$= \frac{n!}{(n-k)!\,k!}$

$= n \cdot \frac{(n-1)!}{k!(n-k)!}$

$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

$k\binom{n}{k} = n\binom{n-1}{k-1}$

$k\binom{n}{k} = n\binom{n-1}{k-1}$ where $\in \mathbb{Z}$

$\Rightarrow n \mid k\binom{n}{k}$ since $gcd(n,k)$, $n \mid \binom{n}{k}$ ☑ proof by direct

5. [8 *marks*]

(a) Determine the prime factorization of (i) 180 and (ii) 252.

i) $180 = 2^2 \cdot 3^2 \cdot 5$

✓ ok, but show some work to justify answer!

ii) $252 = 2^2 \cdot 3^2 \cdot 7$

ex $180 = 2 \times 90$
$= 2 \times 9 \times 10$
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$252 = 2 \times 126$
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8/8

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$[180, 252] = \frac{180 \cdot 252}{gcd(180,252)}$ or $2^2 \cdot 3^2 \cdot 5 \cdot 7 = 1260$ ✓

(c) Apply the Euclidean Algorithm to determine integers k and l such that $(180, 252) = 180k + 252l$.

$252 = 1 \cdot 180 + 72$ $\qquad 72 = 252 - 180$

$180 = 2 \cdot 72 + 36$ ✓ $\qquad 36 = 180 - 2 \cdot 72$

$72 = 2 \cdot 36 + 0$

$36 = (180, 252) = 180 - 2 \cdot 72$ ✓

$= 180 - 2 \cdot (252 - 180)$

$= 3 \cdot 180 - 2 \cdot 252$

$\boxed{k=3}$ $\boxed{l=-2}$ ✓

6. [12 *marks*]

(a) Prove any <u>one</u> of the following statements:

1. If $a, b \in \mathbb{Z}$ and $a \equiv b \pmod{n}$, then $ac \equiv bc \pmod{n}$ for any $c \in \mathbb{Z}$.
2. If $a, b \in \mathbb{Z}$ with $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.
3. If $a, b \in \mathbb{Z}$ and $a \equiv b \pmod{n}$, then $a^2 \equiv b^2 \pmod{n}$.

1) Assume $a, b \in \mathbb{Z}$ and $a \equiv b (\text{mod } n)$.

$ac - bd$

$a - b = kn$, $k \in \mathbb{Z}$

$\Rightarrow ac - bc = (ck)n$ where $ck \in \mathbb{Z}$,

$a - b = kn$

$c - d = ln$

$\Rightarrow ac \equiv bc \ (\text{mod } n)$

3) Assume $a, b \in \mathbb{Z}$ $a \equiv b (\text{mod } n)$

$\Rightarrow a - b = n \cdot q, q \in \mathbb{Z}$

$\{ a^2 - b^2 = n \cdot (\mathbb{Z})$

$a^2 - b^2 = n\,(q(a+b))$

2) Assume $a, b \in \mathbb{Z}$ $a \equiv b (\text{mod } n)$, $c \equiv d (\text{mod } n)$

$a - b = n \cdot q$

$c - d = n \cdot r$

(b) Use the statements given in part(a) to determine the remainder when 2^{37} is divided by 7.

$5 \equiv 2 (\text{mod } 3)$

$6 \equiv 0 (\text{mod } 3)$

$2^{12} \equiv 1 (\text{mod } 7)$ $\quad a \equiv b$

$2^{37} \equiv x \ (\text{mod } 7)$ $\quad 2^{24} \equiv 1 (\text{mod } 7)$ $\quad c \equiv d$

$\cancel{8} \equiv \cancel{1} \ (\text{mod } 7)$ $\quad 2^{36} \equiv 1 \ (\text{mod } 7)$ (by (2)) $\quad ac \equiv bd$

$2^3 \equiv 1 \ (\text{mod } 7)$ $\quad 2^{37} \equiv 2 \ (\text{mod } 7)$

$(2^3)^2 \equiv 1^2 (\text{mod } 7)$ (3) $\quad \therefore R = 2$

$2^6 \equiv 1 \ (\text{mod } 7)$

$2^{24} \equiv 1 \ (\text{mod } 7)$

(c) Solve the congruence: $7x \equiv 41 \pmod{13}$.

$41 = 7x + 13p$ for $p \in \mathbb{Z}$

$28 = 7x$

$x = 4$

1/3

→ solution will be an equivalence class.

$7x - 41 = 13n$, $n \in \mathbb{Z}$

$\Rightarrow ac-bc=(ck)n$ where $ck\in\mathbb{Z}$, $a-b=kn$

$c-d=ln$

$\Rightarrow ac\equiv bc \pmod n$

3) Assume $a,b\in\mathbb{Z}$ $a\equiv b \pmod n$

$\Rightarrow a-b=n\cdot q, q\in\mathbb{Z}$

$a^2-b^2=n(q(a+b))$

$\{ a^2-b^2=n\cdot(\mathbb{Z})$

2) Assume $a,b\in\mathbb{Z}$ $a\equiv b \pmod n$, $c\equiv d \pmod n$

$a-b=n\cdot q$

$c-d=n\cdot r$

(b) Use the statements given in part(a) to determine the remainder when 2^{37} is divided by 7.

$5\equiv 2 \pmod 3$

$6\equiv 0 \pmod 3$

$2^{37}\equiv x \pmod 7$

$8\equiv 1 \pmod 7$

$2^3\equiv 1 \pmod 7$

$(2^3)^2\equiv 1^2 \pmod 7$ (3)

$2^6\equiv 1 \pmod 7$

$2^{24}\equiv 1 \pmod 7$

$2^{12}\equiv 1 \pmod 7$ $a\equiv b$

$2^{24}\equiv 1 \pmod 7$ $c\equiv d$

$2^{36}\equiv 1 \pmod 7$ (by (2)) $ac\equiv bd$

$2^{37}\equiv 2 \pmod 7$

$\boxed{\therefore R=2}$

(c) Solve the congruence: $7x\equiv 41 \pmod{13}$.

$41=7x+13p$ for $p\in\mathbb{Z}$

$28=7x$

1/3

$x=4$

→ solution will be an equivalence class.

$7x-41=13n$, $n\in\mathbb{Z}$

$41-13$

$7x-2=13n$

28

15

$7x-13n=2$

2

$7x-13n=\gcd(7,13)=1$

$13=7(1)+6$

$7=6(1)+1$

$x_0=4$, $n_0=2$ is one solution

$1=7-6$

and $x=4+13j$, $j\in\mathbb{Z}$

$1=7(2)-13$

7. [~~6~~ 5 *marks*] Use the Binomial Theorem to find the term containing x^{18} in the expansion of: $(x^2 - 2y)^{12}$.

$$(x^2-2y)^{12} = \sum_{k=0}^{12} \binom{12}{k} (x^2)^{12-k} (-2y)^k$$ ✓

$$x^{18} = (x^2)^{12-k}$$
$$18 = 2\cdot(12-k)$$ ✓
$$9 = 12-k$$
$$k=3$$

5/5

$$\binom{12}{3} = \frac{12!}{9!\,3!} = \frac{12\cdot 11\cdot 10}{3\cdot 2} = 2\cdot 11\cdot 10 = 220$$

$$(-2y)^3 = -8y^3$$

$$\boxed{-220\,x^{18}y^3}$$ ✓

8. [~~7~~ 8 *marks*]

(a) Evaluate the following, expressing your final answer in standard form: $\overline{|24-7i| - 2i\,(1+3i)}$

$$|24-7i| = \sqrt{24^2+7^2} = \sqrt{576+49} = \sqrt{625} = 25$$ ✓

$$\begin{array}{r} 24 \\ \times 24 \\ \hline 480 \\ 96 \\ \hline 576 \end{array}$$

3/3

$$-2i(1+3i) = -2i-6i^2 = -2i+6$$ ✓

$$25 - 2i + 6 = 31 - 2i$$

$$\overline{31-2i} = \boxed{31+2i}$$ ✓

(b) Let $z, w \in \mathbb{C}$. Prove that if $|z| = |w|$, then $\dfrac{z+w}{z-w}$ is pure imaginary.

Let $z = a+bi \quad a,b \in \mathbb{R}$
$w = c+di \quad c,d \in \mathbb{R}$

$|z| = \sqrt{a^2+b^2}$
$|w| = \sqrt{c^2+d^2}$ ✓

$|z| = |w|$
$\Leftrightarrow \sqrt{a^2+b^2}\ \sqrt{c^2+d^2}$

$$\frac{z+w}{z-w} = \frac{a+bi+c+di}{a+bi-(c+di)} = \frac{(a+c)+(b+d)i}{(a-c)+(b-d)i} \times \frac{(a+c)-(b+d)i}{(a-c)-(b+d)i}$$ ✓

$18 = 2\cdot(12-k)$ ✓

$9 = 12-k$

$k = 3$

5/5

$\binom{12}{3} = \frac{12!}{9!\,3!} = \frac{12\cdot 11\cdot 10}{3\cdot 2}$

$= 2\cdot 11\cdot 10$

$= 220$

$(-2y)^3 = -8y^3$

$\boxed{-220\,x^{18}y^3}$ ✓

8. [~~7~~ 8 *marks*]

(a) Evaluate the following, expressing your final answer in standard form: $\overline{|24-7i| - 2i\,(1+3i)}$

$|24-7i| = \sqrt{24^2+7^2}$

$= \sqrt{576+49}$

$= \sqrt{625}$ ✓

$= 25$

$24 \times 24 = 480 + 96 = 576$

3/3

$-2i(1+3i)$

$= -2i-6i^2$ ✓

$= -2i+6$

$25-2i+6$

$= 31-2i$

$\overline{31-2i} = \boxed{31+2i}$ ✓

(b) Let $z, w \in \mathbb{C}$. Prove that if $|z| = |w|$, then $\dfrac{z+w}{z-w}$ is pure imaginary.

Let $z = a+bi$ $a,b \in R$

$w = c+di$ $c,d \in R$

$|z| = \sqrt{a^2+b^2}$

$|w| = \sqrt{c^2+d^2}$ ✓

$|z| = |w|$

$\Leftrightarrow \sqrt{a^2+b^2}\ \sqrt{c^2+d^2}$

$a^2+b^2 = c^2+d^2$

5/5

$\frac{z+w}{z-w}$

$= \frac{a+bi+c+di}{a+bi-(c+di)}$

$= \frac{(a+c)+(b+d)i}{(a-c)+(b-d)i} \times \frac{(a-c)-(b-d)i}{(a-c)-(b-d)i}$ ✓

$= \frac{(a^2-c^2)-(a+c)(b-d)i+(a-c)(b+d)i-(b^2-d^2)}{(a-c)^2-(b-d)^2}$ ✓

$\frac{a^2+b^2-c^2-d^2+[(a-c)(b+d)-(a+c)(b-d)]i}{a^2-2ac+c^2-b^2+2bd-d^2}$

Since

$a^2+b^2 = c^2+d^2$

$a^2+b^2-c^2\ \overline{+}\ d^2 = 0$

careful!

$= \frac{[(a-c)(b+d)-(a+c)(b-d)]\,i}{a^2-2ac+c^2-b^2+2bd-d^2}$

where $\in R$

9. [6 *marks*] Use polar form, or exponential form, and DeMoivre's Theorem to evaluate $\dfrac{(1+i)^3}{(1-\sqrt{3}i)^9}$.

Express your final answer in standard form.

$\theta = \frac{\pi}{4}$

$(1+i) = r\cos\theta + r\sin\theta\, i$

$= \sqrt{2}\cos\frac{\pi}{4} \cdot \sqrt{2}\sin\frac{\pi}{4}\, i$

$= \sqrt{2}\,\text{cis}\frac{\pi}{4}$ ✓

$(1+i)^3 = (\sqrt{2})^3\,\text{cis}(\frac{\pi}{4}\cdot 3)$

$= 2\sqrt{2}\,\text{cis}\frac{3\pi}{4}$

4/6

$\dfrac{(1+i)^3}{(1-\sqrt{3}i)^9}$

$= \dfrac{2\sqrt{2}\,\text{cis}\frac{3\pi}{4}}{512\,\text{cis}\pi}$ ✓

$\boxed{= \dfrac{\sqrt{2}}{256}\,\text{cis}\frac{7\pi}{4}}$

$=$

$(1-\sqrt{3}i) = 2\cos\frac{\pi}{3} + 2\sin\frac{\pi}{3}\, i$

$= 2\,\text{cis}\frac{\pi}{3}$

$(1-\sqrt{3}i)^9 = 2^9\,\text{cis}(\frac{\pi}{3}\cdot 9)$

$= 512\,\text{cis}3\pi$

or

$512\,\text{cis}\,\pi$

10. [8 *marks*] Find the solution set to the equation $x^4 + 16 = 0$, given that:

(i) $x \in \mathbb{R}$, the set of real numbers; (ii) $x \in \mathbb{C}$, the set of complex numbers.

i) $x^4 = -16$

No solution for $x \in \mathbb{R}$ ✓ $\therefore$ sol'n set $= \phi$

$-16 = (4i)^2$

$(-4i)^2$

ii) $x^4 = -16$

$x^4 = -16 + 0i = 16\,\text{cis}(\pi)$

4/8

$16^{\frac{1}{4}}\,\text{cis}\left(\frac{0 + 2\pi k}{4}\right)$ $k = 0, 1, 2, 3$

$4i =$

$\cos\theta = 0$

$\sin\theta = 1$

$-16 = (4i)^2$

$\neq (-4i)^2$ → do same thing for this part

$-4i = (?)^2$

$4^{\frac{1}{2}}\,\text{cis}\left(\frac{\frac{3\pi}{2} + 2\pi k}{2}\right)$ $k = 0, 1$

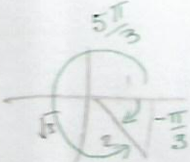
$= \frac{1}{\sqrt{2}} \operatorname{cis}\frac{\pi}{4}$ $\qquad = \frac{\;}{512 \operatorname{cis}\pi}$

4/6

$$(1+i)^3 = (\sqrt{2})^3 \operatorname{cis}\left(\frac{\pi}{4}\cdot 3\right)$$

$$= 2\sqrt{2} \operatorname{cis}\frac{3\pi}{4}$$

$$\boxed{= \frac{\sqrt{2}}{256} \operatorname{cis}\frac{7\pi}{4}}$$



$$(1-\sqrt{3}i) = 2\cos\left(\frac{\pi}{3}\right) + 2\sin\left(\frac{\pi}{3}\right) i$$

$$= 2 \operatorname{cis}\frac{\pi}{3}$$

$$(1-\sqrt{3}i)^9 = 2^9 \operatorname{cis}\left(\frac{\pi}{3}\cdot 9\right)$$

$$= 512 \operatorname{cis} 3\pi$$

or

$$512 \operatorname{cis}\pi$$

10. [8 *marks*] Find the solution set to the equation $x^4 + 16 = 0$, given that:

(i) $x \in \mathbb{R}$, the set of real numbers; (ii) $x \in \mathbb{C}$, the set of complex numbers.

i) $x^4 = -16$

No solution for $x \in \mathbb{R}$ $\therefore$ sol'n set $= \emptyset$

$-16 = (4i)^2$

ii) $x^4 = -16$

$(-4i)^2$

$x^4 = -16 + 0i = 16 \operatorname{cis}(\pi)$

4/8

$$16^{\frac{1}{4}} \operatorname{cis}\left(\frac{0 + 2\pi k}{4}\right) \quad k = 0, 1, 2, 3$$

$4i =$ $\cos\theta = 0$

$\sin\theta = 1$

$-16 = (4i)^2$

$= (-4i)^2$ ⟶ do same thing for this part

$-4i = (?)^2$

$$4^{\frac{1}{2}} \operatorname{cis}\left(\frac{\frac{\pi}{2} + 2\pi k}{2}\right) \quad k = 0, 1$$

$k=0 \Rightarrow 2\cos\frac{\pi}{4} + 2\sin\frac{\pi}{4} i = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$

$k=1 \Rightarrow 2\cos\frac{5\pi}{4} + 2\sin\frac{5\pi}{4} i = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$

$x = \left\{ \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i, \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i, -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i \right\}$

show where these 2 ans. come from.