

58
80

Student Number: 140589590

[10 marks] 1. Consider the following data among 110 students in the MA121 class:

A 30 students are also taking MA122

B 35 students are also taking MA103

$A \cap B$ 20 students are taking both MA122 and MA103

(a) Fill the blanks:

Let U be the set that consists of students in the MA121 class. The number of elements in U is $|U| = 110$.

The set $A = \{x \in U : x \text{ studies MA122}\}$.

The set $B = \{x \in U : x \text{ studies MA103}\}$ has 35 elements.

The students of MA121 who are also taking both MA103 and MA122 form the set $A \cap B$.

In parts (b), (c) and (d) below, fill the blanks with numbers and circle the correct sets. The symbols of sets are introduced in part (a) above.

(b) The number of students in this class who are also taking either MA122 or MA103 is

(i) 45

(ii) 65

(iii) 25

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 30 + 35 - 20$$

They form the set

(i) $A \cap B$

(ii) $A \cup B$

(iii) $A \Delta B$

(iv) $U - A \cup B$

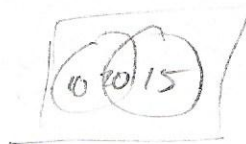
$\frac{10}{10}$

(c) The number of students in this class who are also taking exactly one of MA122 and MA103 is

(i) 45

(ii) 65

(iii) 25



They form the set

(i) $A \cap B$

(ii) $A \cup B$

(iii) $A \Delta B$

(iv) $U - A \cup B$

(d) The number of students in this class who are taking neither MA122 nor MA103 is

(i) 45

ANS 20 students are taking both MA122 and MA103

(a) Fill the blanks:

Let U be the set that consists of students in the MA121 class. The number of elements in U is $|U| = 110$.

The set $A = \{x \in U : x \text{ studies MA122}\}$.

The set $B = \{x \in U : x \text{ studies MA103}\}$ has 35 elements.

The students of MA121 who are also taking both MA103 and MA122 form the set $A \cap B$.

In parts (b), (c) and (d) below, fill the blanks with numbers and circle the correct sets. The symbols of sets are introduced in part (a) above.

(b) The number of students in this class who are also taking either MA122 or MA103 is

(i) 45

(ii) 65

(iii) 25

$$|A \cup B| = |A| + |B| - |A \cap B| \\ = 30 + 35 - 20$$

They form the set

(i) $A \cap B$

(ii) $A \cup B$

(iii) $A \Delta B$

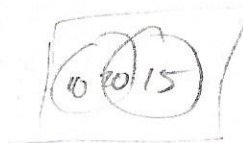
(iv) $U - A \cup B$

(c) The number of students in this class who are also taking exactly one of MA122 and MA103 is

(i) 45

(ii) 65

(iii) 25



They form the set

(i) $A \cap B$

(ii) $A \cup B$

(iii) $A \Delta B$

(iv) $U - A \cup B$

(d) The number of students in this class who are taking neither MA122 nor MA103 is

(i) 45

(ii) 65

(iii) 25

$$110 - 45 = 65$$

They form the set

(i) $A \cap B$

(ii) $A \cup B$

(iii) $A \Delta B$

(iv) $U - A \cup B$

$\frac{10}{10}$

[10 marks] 2. (a) Use the Principle of **Mathematical Induction** to prove that, for all natural numbers $n \geq 3$,

$$\binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \dots + \binom{n}{n-3} = \binom{n+1}{n-3}$$

[Hint: The binomial coefficient formula $\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}$ should be used somewhere in your proof.]

1) **Base Case**. Show $n=3$ true. $\forall n \in \mathbb{N}$

L.S. $\binom{3}{3-3} = \binom{3}{0} = 1$

R.S. $\binom{3+1}{3-3} = \binom{4}{0} = 1$ L.S. = R.S. \therefore True.

2) **Inductive Hypothesis**. Assume $n=k+3$ true. $\forall k+3 \in \mathbb{N}$

$\binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \dots + \binom{k+3}{k+3-3} = \binom{k+3+1}{k+3-3}$

(10)
(10)

3) **Induction Step**. Show $n=k+4$ true. $\forall k+4 \in \mathbb{N}$

$\binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \dots + \binom{k+4}{k+4-3} = \binom{k+4+1}{k+4-3}$

$\downarrow \binom{k+3}{k}$ $\downarrow \binom{k+4}{k}$

$\downarrow \binom{k+4}{k+1}$ $\downarrow \binom{k+5}{k+1}$

From IH.

$$\begin{aligned} & \binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \dots + \binom{k+3}{k} + \binom{k+4}{k+1} \\ &= \binom{k+3+1}{k+3-3} + \binom{k+4}{k+1} \\ &= \binom{k+4}{k} + \binom{k+4}{k+1} \\ &= \binom{k+5}{k+1} \quad \star \text{From theorem 3.3.2} \\ &= \binom{(k+4)+1}{(k+4)-3} = \text{R.S.} \quad \therefore \text{True for I.S.} \end{aligned}$$

Proved by mathematical induction that it is True

[4 marks] (b) Write the sum in the formula of part (a) in **sigma notation**:

4
4

$\sum_{i=3}^n \binom{i}{i-3} = \binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \dots + \binom{n}{n-3}$

[5 marks] 3. The course MA121 has two sections this semester. There are 168 students in Section A and 199 students in Section B.

(a) Show that at least two students who study MA121 this semester have the same birthday.

3 168 199 367 367! 367! 367.366

[Hint: The binomial coefficient formula $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ should be used somewhere in your proof.]

1) Base Case. Show $n=3$ true. for $n \in \mathbb{N}$

L.S. $\binom{3}{3-3} = \binom{3}{0} = 1$
 R.S. $\binom{3+1}{3-3} = \binom{4}{0} = 1$ L.S. = R.S. \therefore True.

2) Inductive Hypothesis. Assume $n=k+3$ true. for $k+3 \in \mathbb{N}$

$\binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \dots + \binom{k+3}{k+3-3} = \binom{k+3+1}{k+3-3}$

3) Inductive Step. Show $n=k+4$ true. for $k+4 \in \mathbb{N}$

$\binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \dots + \binom{k+4}{k+4-3} = \binom{k+4+1}{k+4-3}$

From IH.

$$\begin{aligned} & \binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \dots + \binom{k+3}{k} + \binom{k+4}{k+1} \\ &= \binom{k+3+1}{k+3-3} + \binom{k+4}{k+1} \\ &= \binom{k+4}{k} + \binom{k+4}{k+1} \\ &= \binom{k+5}{k+1} \quad \star \text{ From theorem 5.3.2} \\ &= \binom{(k+4)+1}{(k+4)-3} = \text{R.S.} \quad \therefore \text{True for I.S.} \end{aligned}$$

Proved by induction that it is True

[4 marks]

(b) Write the sum in the formula of part (a) in sigma notation:

$\sum_{i=3}^n \binom{i}{i-3} = \binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \dots + \binom{n}{n-3}$

[5 marks]

3. The course MA121 has two sections this semester. There are 168 students in Section A and 199 students in Section B.

(a) Show that at least two students who study MA121 this semester have the same birthday.

$$\begin{array}{r} 3 \ 168 \\ \underline{199} \\ 5 \ 367 \end{array}$$

~~$\binom{367}{2} = \frac{367!}{(367-2)! \cdot 2!} = \frac{367!}{365! \cdot 2!} = \frac{367 \cdot 366}{2} = 367 \cdot 183 > 365$~~

1
 less year - 2

(b) Which method do you use in part (a) above? Circle your answer.

- A. Addition Principle
- B. Pigeonhole Principle
- C. Multiplication Principle
- D. Mathematical Induction

4. Prove or disprove the following statements. In each case, indicate first which method from the list here you are using:

- 1) Direct proof of implication
- 2) Proof of the contrapositive of implication
- 3) Indirect proof by contradiction
- 4) Non-constructive proof of existence
- 5) Proof of existence by construction
- 6) Disproof by counterexample

[5 marks]

(a) Let z and w be complex numbers. If $z \notin \mathbb{R}$ and $w \notin \mathbb{R}$, then $|z + w| < |z| + |w|$.

The method: 6

Counterexample:

let $z, w \in \mathbb{C}$, $z, w \notin \mathbb{R}$

$$z = 2 + 2i$$

$$w = 2 + 2i$$

$$z + w = 4 + 4i$$

$$|z + w| = \sqrt{4^2 + 4^2}$$

$$= 4\sqrt{2}$$

$$|z| = \sqrt{2^2 + 2^2}$$

$$= \sqrt{2} \sqrt{2}$$

$$|w| = \sqrt{2^2 + 2^2}$$

$$= 2\sqrt{2}$$

$$4\sqrt{2} = 2\sqrt{2} + 2\sqrt{2}$$

$$= 2(2\sqrt{2}) = 4\sqrt{2}$$

$$|z + w| = |z| + |w|$$

Stat is false

Disproved by counterexample.

\square

[5 marks]

(b) If Let A and B be nonempty finite sets with $|A| = |B| = n$ and let $f : A \rightarrow B$ be a function. If f is a one-to-one function, then it is also an onto function.

The method: 1

let $f : A \rightarrow B$

$$f(A) = n$$

$$f(B) = n$$

$$f(A) = f(B)$$

$$|A| = |B| = n$$

\therefore it is one-to-one

Prove onto: $f(x) = y$ for all x in domain mapped into y -codomain

$$\Leftrightarrow f^{-1}(n) = |A|$$

$$\Leftrightarrow n = f^{-1}(n)$$

$$\Leftrightarrow f(n) = |B|$$

$$\Leftrightarrow f(f^{-1}(n)) = |B|$$

$$\Leftrightarrow |A| = |B|$$

\therefore it's onto

Statement is true, proved by direct

\therefore If every element of A maps into one distinct element of B , and the number of elements of A and B are equal, then A is mapped onto B .

can't do this

need more

[5 marks]

(c) Denote by $|X|$ the number of elements in a finite set X . Let A and B be finite sets. Then $|A - B| + |B - A| = |A \cup B|$.

The method: 6

Counterexample

Let A and B be finite sets.

$$A = \{1, 2, 3, 4\}$$

$$B = \{3, 4, 5\}$$

$$A - B = \{1, 2, 4\}$$

$$|A - B| = 3$$

$$B - A = \{5\}$$

$$|B - A| = 1$$

$$|A - B| + |B - A| = 3 + 1 = 4$$

$$A \cup B = \{1, 2, 3, 4, 5\}$$

$$|A \cup B| = 5$$

$$4 \neq 5$$

$$|A - B| + |B - A| \neq |A \cup B|$$

Statement is false

disproved by counterexample

\square

[5 marks]

5 (d) Let z and w be complex numbers. Then $w\bar{z} + \bar{w}z$ is a real number.

The method: 1

- 3) Indirect proof by contradiction
 4) Non-constructive proof of existence
 5) Proof of existence by construction
 6) Disproof by counterexample

[5 marks]

(a) Let z and w be complex numbers. If $z \notin \mathbb{R}$ and $w \notin \mathbb{R}$, then $|z+w| < |z| + |w|$.

The method: 6

Counterexample:

Let $z, w \in \mathbb{C}$, $z, w \notin \mathbb{R}$

$$z = 2 + 2i$$

$$w = 2 + 2i$$

$$z+w = 4 + 4i$$

$$|z+w| = \sqrt{4^2 + 4^2}$$

$$= 4\sqrt{2}$$

$$|z| = \sqrt{2^2 + 2^2}$$

$$= 2\sqrt{2}$$

$$|w| = \sqrt{2^2 + 2^2}$$

$$= 2\sqrt{2}$$

$$4\sqrt{2} = 2\sqrt{2} + 2\sqrt{2} = 2(2\sqrt{2}) = 4\sqrt{2}$$

$$|z+w| = |z| + |w|$$

Stat is false

∴ disproved by counterexample. \square

[5 marks]

(b) Let A and B be nonempty finite sets with $|A| = |B| = n$ and let $f : A \rightarrow B$ be a function. If f is a one-to-one function, then it is also an onto function.

The method: 1

Let $f : A \rightarrow B$

$$f(A) = n$$

$$f(B) = n$$

$$f(A) = f(B)$$

$$|A| = |B| = n$$

∴ it is one-to-one

Prove onto: $f(x) = y$ for all x in domain mapped into y -value

$$\Leftrightarrow f(A) = B$$

$$\Leftrightarrow n = f(A)$$

$$\Leftrightarrow f(A) = |B|$$

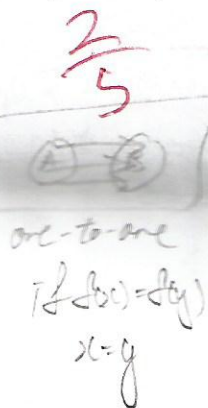
$$\Leftrightarrow f(A) = |B|$$

$$\Leftrightarrow |A| = |B|$$

∴ it's onto

∴ Statement is True, proved by direct.

∴ If every element of A maps into one distinct element of B , and the number of elements of A and B are equal, then A is mapped onto B .



[5 marks]

(c) Denote by $|X|$ the number of elements in a finite set X . Let A and B be finite sets. Then $|A-B| + |B-A| = |A \cup B|$.

The method: 6

Counterexample

Let A and B be finite sets.

$$A = \{1, 2, 3, 4\}$$

$$B = \{3, 4, 5\}$$

$$A-B = \{1, 2, 4\}$$

$$|A-B| = 3$$

$$B-A = \{5\}$$

$$|B-A| = 1$$

$$|A-B| + |B-A| = 3 + 1 = 4$$

$$A \cup B = \{1, 2, 3, 4, 5\}$$

$$|A \cup B| = 5$$

$$4 \neq 5$$

$$|A-B| + |B-A| \neq |A \cup B|$$

∴ Statement is false

disproved by counterexample. \square

[5 marks]

5 (d) Let z and w be complex numbers. Then $w\bar{z} + \bar{w}z$ is a real number.

The method: 1

Let $z, w \in \mathbb{C}$

$$z = a+bi \text{ for } a, b \in \mathbb{R}$$

$$w = c+di \text{ for } c, d \in \mathbb{R}$$

$$\bar{z} = a-bi$$

$$\bar{w} = c-di$$

$$w\bar{z} + \bar{w}z = (c+di)(a-bi) + (c-di)(a+bi)$$

$$= ac - bci + adi - bdi^2 + ac + cbi - adi - bdi^2$$

$$= ac + ac - bci + bci + adi - adi - bdi^2 - bdi^2$$

$$= 2ac - (bd)i^2$$

$$= 2ac + 2bd$$

$$= 2(ac + bd)$$



5
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[9 marks] 5. Find the 4th term in the expansion of $(x^2 - \frac{1}{x})^{11}$. Show all steps of your work.

4th term = k=3

$a = x^2$
 $b = \frac{1}{x}$

$$\binom{11}{3} = \frac{11!}{(11-3)! 3!}$$

$$= \frac{11!}{8! \cdot 6}$$

$$= \frac{11 \cdot 10 \cdot 9}{6}$$

$$= 165$$

$\binom{n}{k} a^{n-k} b^k$

$= \binom{11}{3} (x^2)^{11-3} (\frac{1}{x})^3$

$= 165 \cdot (x^2)^8 (\frac{1}{x^3})$

$= -165 \cdot x^{16} \cdot \frac{1}{x^3} = \boxed{-165x^{13}}$

[13 marks] 6. Let A and B be subsets of the universal set U with $A \subseteq B$. Denote by B^c the complement of B in U. Prove that, as subsets of the product set $U \times U$,

$x \in B$

$[B \times (A \cup B^c)] \cap [(A \cup B^c) \times B] = A \times A$

$y \in A \cup B^c$

[Hint: You may start with "If the ordered pair $(x, y) \in [B \times (A \cup B^c)] \cap [(A \cup B^c) \times B]$ "]

If the ordered pair $(x, y) \in [B \times (A \cup B^c)] \cap [(A \cup B^c) \times B]$

LS $\Leftrightarrow (x, y) \in [B \times (A \cup B^c)]$ and $(x, y) \in [(A \cup B^c) \times B]$

$\Leftrightarrow [x \in B \text{ and } y \in (A \cup B^c)] \text{ and } [y \in (A \cup B^c) \text{ and } x \in B]$

$\Leftrightarrow x \in B \text{ and } y \in A \cup B^c \text{ or } y \in A \cup B^c \text{ and } x \in B$

$\Leftrightarrow (x \in B \text{ and } (y \in A \text{ or } y \in B^c)) \text{ or } (y \in A \text{ or } y \in B^c \text{ and } x \in B)$

$\Leftrightarrow x \in B \text{ or } x \in B \Rightarrow x \in B \cup B^c \Rightarrow x \in U$

$\Leftrightarrow (x \in U \text{ and } y \in A) \text{ or } (x \in U \text{ and } y \in A)$

$\Leftrightarrow x \in U \text{ and } (y \in A \text{ or } y \in A)$

$\Leftrightarrow x \in U \text{ and } (y \in A \cup A)$

$\Leftrightarrow x \in U \text{ and } y \in U$

$\Leftrightarrow (x, y) \in U \times U$

$\Leftrightarrow (x, y) \in (A \times A)$

as $A \subseteq B \subseteq U$

P.S. $\rightarrow A \times A \subseteq U \times U$

$A \times A \subseteq [B \times (A \cup B^c)] \cap [(A \cup B^c) \times B]$

$\forall (x, y) \in A \times A = [B \times (A \cup B^c)] \cap [(A \cup B^c) \times B]$

The statement is true proved by direct proof

[9 marks] 7. Simplify $\frac{1+i}{3i} - \frac{1+i}{1-i}$. Express your answer in standard form, $a + bi$.

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$$= \binom{11}{3} (x^2)^{11-3} \left(\frac{-1}{x}\right)^3$$

$$= 165 \cdot (x^2)^8 \left(\frac{-1}{x}\right)$$

$$= -165 \cdot x^{16} \cdot \frac{1}{x^3} = \boxed{-165x^{13}}$$

$$= \frac{11 \cdot 10 \cdot 9}{6 \cdot 2 \cdot 1}$$

$$= 165$$

[13 marks] 6. Let A and B be subsets of the universal set U with $A \subseteq B$. Denote by B^c the complement of B in U . Prove that, as subsets of the product set $U \times U$,

$$x \in B, y \in A \cup B^c \implies [B \times (A \cup B^c)] \cap [(A \cup B^c) \times B] = A \times A$$

[Hint: You may start with "If the ordered pair $(x, y) \in [B \times (A \cup B^c)] \cap [(A \cup B^c) \times B]$ "]

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LS $\iff (x, y) \in [B \times (A \cup B^c)]$ and $(x, y) \in [(A \cup B^c) \times B]$

$\iff [x \in B \text{ and } y \notin (A \cup B^c) \text{ or } y \in A \cup B^c \text{ and } x \in B]$ and $[y \in (A \cup B^c) \text{ and } x \in B \text{ or } x \in B \text{ and } y \notin (A \cup B^c)]$

$\iff x \in B$ and $y \notin A \cup B^c$ or $y \in A \cup B^c$ and $x \in B$ *same they*

$\iff (x \in B \text{ and } (y \notin A \text{ or } x \in B)) \text{ or } (y \in A \text{ or } x \in B) \text{ and } x \in B$

$\iff x \in B \text{ or } x \in B \implies x \in B \cup B^c \implies x \in U$

$\iff (x \in U \text{ and } y \in A) \text{ or } (x \in U \text{ and } y \in A)$

$\iff x \in U \text{ and } (y \in A \text{ or } y \in A)$

$\iff x \in U \text{ and } (y \in A \cup A)$

$\iff x \in U \text{ and } y \in U$

$\iff (x, y) \in U \times U$

$\iff (x, y) \in (A \times A)$ **P.S.**

as $A \subseteq B \subseteq U$

$A \times A \subseteq [B \times (A \cup B^c)] \cap [(A \cup B^c) \times B]$

$\forall (x, y) \in A \times A \implies (x, y) \in [B \times (A \cup B^c)] \cap [(A \cup B^c) \times B]$

The statement is true proved by direct proof

[9 marks] 7. Simplify $\frac{1+i}{3i} - \frac{1+i}{1-i}$. Express your answer in standard form, $a + bi$.

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$$\frac{1+i}{3i} - \frac{1+i}{1-i}$$

$$= \frac{(1+i)(3i)}{(3i)(3i)} - \frac{(1+i)(1+i)}{(1-i)(1+i)}$$

$$= \frac{3i + 3i^2}{9i^2} - \frac{1 + 2i + i^2}{1 - i^2}$$

$$= \frac{3i - 3}{-9} - \frac{1 + 2i - 1}{2}$$

$$\frac{1}{3} - \frac{4}{3}i$$

$$= \frac{1}{3} + \left(-\frac{4}{3}\right)i$$

$$a = \frac{1}{3}$$

$$b = -\frac{4}{3}$$