63/80

**Student Number:** 140587590

[*4 marks*] 1. (a) Complete the truth table of the compound statement with the the disjunctive normal form $(p\wedge q\wedge r)\vee(p\wedge q\wedge\sim r)\vee(p\wedge\sim q\wedge r)\vee(\sim p\wedge q\wedge r)\vee(\sim p\wedge q\wedge\sim r)\vee(\sim p\wedge\sim q\wedge r)\vee(\sim p\wedge\sim q\wedge\sim r)$.

4/4

| $p$ | $q$ | $r$ | statement |
|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | T |
| $T$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | F |
| $F$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | T |
| $F$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | T |

✓

$(p\wedge q\wedge r)\vee(p\wedge q\wedge\sim r)\vee(p\wedge\sim q\wedge r)\vee(\sim p\wedge q\wedge r)\vee(\sim p\wedge q\wedge\sim r)$

T T T T T

$\vee(\sim p\wedge\sim q\wedge r)\vee(\sim p\wedge\sim q\wedge\sim r)$

T T

[*2 marks*] (b) Find a compound statement, equivalent to the one in part (a), that has a simplest expression in which each of $p$, $q$ and $r$ appears **only once**.

1/2

**Answer:** $p\to(q\overset{\times}{\wedge}r)$

[*7 marks*] 2. (a) Complete the following truth tables.

7/7

| $p$ | $q$ | $q\to p$ | $\sim p\vee q$ | $p\wedge\sim q$ | $\sim p\to\sim q$ | $\sim(p\to q)$ | $p\to q$ | $\sim(q\to p)$ |
|---|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $F$ | $T$ | F | $T$ | F |
| $T$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ | F | $F$ |
| $F$ | $T$ | F | $T$ | F | $F$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | T | $F$ | T | $F$ | $T$ | $F$ |

✓

[*4 marks*] (b) According to the truth tables above in part (a), determine whether each of the following statements is TRUE or FALSE. Circle your answer.

4/4

a) $(q\to p)\Leftrightarrow(p\wedge\sim q)$ (TRUE or [✓ FALSE]) ✓ b) $(\sim p\to\sim q)\Leftrightarrow(q\to p)$ ([✓ TRUE] or FALSE)

c) $\sim(q\to p)\Leftrightarrow(p\to q)$ (TRUE or [✓ FALSE]) d) $\sim(q\to p)\Leftrightarrow(\sim p\vee q)$ (TRUE or [✓ FALSE])

[*9 marks*] 3. Let $S$ be the statement:

*Every integer that is a multiple of 3 is an even number.*

Complete the following sentences.

5/9

(a) $S$ is a False ✓ (TRUE or FALSE) statement.

(b) The **converse** of the statement $S$ is: ✓

All even number is an integer that is a multiple of 3,

which is a False ✓ (TRUE or FALSE) statement.

(c) The **contrapositive** of $S$ is:

~~For some~~ All odd number is ✗ an integer that is not multiple of 3,

which is a TRUE ok (TRUE or FALSE) statement.

(d) The **contrapositive of the converse** of the statement $S$ is: ✓

4/4

| | | | |
|---|---|---|---|
| T | T | T | T |
| T | T | F | T |
| T | F | T | T |
| T | F | F | F |
| F | T | T | T |
| F | T | F | T |
| F | F | T | T |
| F | F | F | T |

$(p\wedge q\wedge r)\vee(p\wedge q\wedge\sim r)\vee(p\wedge\sim q\wedge r)\vee(\sim p\wedge q\wedge r)\vee(\sim p\wedge q\wedge\sim r)$

T T T T T

$\vee(\sim p\wedge\sim q\wedge r)\vee(\sim p\wedge\sim q\wedge\sim r)$

T T

[2 marks] (b) Find a compound statement, equivalent to the one in part (a), that has a simplest expression in which each of $p$, $q$ and $r$ appears **only once**.

1/2

Answer: $p\to(q\overset{x}{\wedge}r)$

[7 marks] 2. (a) Complete the following truth tables.

7/7

| $p$ | $q$ | $q\to p$ | $\sim p\vee q$ | $p\wedge\sim q$ | $\sim p\to\sim q$ | $\sim(p\to q)$ | $p\to q$ | $\sim(q\to p)$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | F | T | F | T | F |
| T | F | T | F | T | T | T | F | F |
| F | T | F | T | F | F | F | T | T |
| F | F | T | T | F | T | F | T | F |

[4 marks] (b) According to the truth tables above in part (a), determine whether each of the following statements is TRUE or FALSE. Circle your answer.

4/4

a) $(q\to p)\Leftrightarrow(p\wedge\sim q)$ (TRUE or [✓ FALSE]) b) $(\sim p\to\sim q)\Leftrightarrow(q\to p)$ ([✓ TRUE] or FALSE)

c) $\sim(q\to p)\Leftrightarrow(p\to q)$ (TRUE or [✓ FALSE]) d) $\sim(q\to p)\Leftrightarrow(\sim p\vee q)$ (TRUE or [✓ FALSE])

[9 marks] 3. Let $S$ be the statement:

*Every integer that is a multiple of 3 is an even number.*

Complete the following sentences.

5/9

(a) $S$ is a False (TRUE or FALSE) statement.

(b) The **converse** of the statement $S$ is:

All even number is an integer that is a multiple of 3,

which is a False (TRUE or FALSE) statement.

(c) The **contrapositive** of $S$ is:

~~For some~~ All odd number is an integer that is not multiple of 3,

which is a TRUE ok (TRUE or FALSE) statement.

(d) The **contrapositive of the converse** of the statement $S$ is:

For some integer that is a multiple of 3.

(e) The **converse of the contrapositive** of the statement $S$ is

~~For some~~ All integer that is not multiple of 3 is an odd number.

(f) The **negation** of the statement $S$ is equivalent to:

For some integer that is a multiple of 3 is an even number.

(g) The **negation** of the statement $S$ IS NOT (IS or IS NOT) equivalent to the statement:

[*10 marks*] 4. Use the Principle of **Mathematical Induction** to prove that, for all natural numbers $n$,

$$\sum_{t=1}^{n}\frac{1}{t(t+1)} = \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\dots+\frac{1}{(n-1)n}+\frac{1}{n(n+1)} = \frac{n}{n+1}.$$

1) Bases for $n=1$

$L.S. = \frac{1}{1(1+1)} = \frac{1}{2}$ ✓

$R.S. = \frac{1}{1+1} = \frac{1}{2}$

$L.S = R.S.$ ∴ True for $n=1$

2i) Induction Hypothesis

For $n=k$, $k\in\mathbb{N}$

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\dots+\frac{1}{k\times(k+1)} = \frac{k}{k+1}$$ ✓

$\frac{10}{10}$

ii) For $n=k+1$, $k\in\mathbb{N}$

$$\underbrace{\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\dots+\frac{1}{k(k+1)}}_{\text{From IH:}}+\frac{1}{(k+1)(k+1+1)} = \frac{k+1}{k+1+1}$$

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\dots+\frac{1}{k(k+1)}+\left[\frac{1}{(k+1)(k+2)}\right] = \frac{k}{k+1}+\left[\frac{1}{(k+1)(k+2)}\right]$$

$$R.S. = \frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

$$\therefore \frac{1}{1\times 2}+\frac{1}{2\times 3}+\dots+\frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

∴ True for any $k, k+1 \in \mathbb{N}$ ✓

▨

$L.S. = \frac{1}{1(1+1)} = \frac{1}{2}$ ✓

$R.S. = \frac{1}{1+1} = \frac{1}{2}$

$L.S = R.S.$ ∴ True for $n=1$

2 i) Induction Hypothesis

For $n=k,\ k \in \mathbb{N}$

$\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \dots + \frac{1}{k\times(k+1)} = \frac{k}{k+1}$ ✓

$\frac{10}{10}$

ii) For $n=k+1,\ k \in N$

$\underbrace{\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \dots + \frac{1}{k(k+1)}} + \frac{1}{(k+1)(k+1+1)} = \frac{k+1}{k+1+1}$

From IH:

$\frac{1}{1\times2} + \frac{1}{2\times3} + \dots + \frac{1}{k(k+1)} + \left[\frac{1}{(k+1)(k+2)}\right] = \frac{k}{k+1} + \left[\frac{1}{(k+1)(k+2)}\right]$

$R.S. = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$

$= \frac{k(k+2)+1}{(k+1)(k+2)}$ ∴ $\frac{1}{1\times2} + \frac{1}{2\times3} + \dots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$

$= \frac{k^2+2k+1}{(k+1)(k+2)}$ ∴ True for any $k,\ k+1 \in \mathbb{N}$ ✓

$= \frac{(k+1)^2}{(k+1)(k+2)}$

$= \frac{k+1}{k+2}$

▨

[6 marks] 5. Let the universe of discourse (the universal set) for $\epsilon$ and $\delta$ be the positive real numbers. Convert the mathematical statement

*For every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that if* $|x-a| < \delta$ , *then* $|f(x) - f(a)| < \epsilon$.

into the logical expression using **quantifiers** $\forall$ and $\exists$ .

$\frac{3}{6}$

−1 −3

x

~~$[\forall\epsilon\,(\epsilon>0) \land \exists\delta\,(\delta>0)] \rightarrow [(|x-a|<\delta) \rightarrow |f(x)-f(a)|<\epsilon]$~~

$\forall\epsilon\ \exists\delta\ [$

6. **Prove** or **disprove** the following statements. In each case, indicate first **which method** from the list here you are using:

   1) *Direct proof of implication*
   2) *Proof of the contrapositive of implication*
   3) *Indirect proof by contradiction*
   4) *Non-constructive proof of existence*
   5) *Proof of existence by construction*
   6) *Disproof by counterexample*

[*5 marks*] (a) Let $n \in \mathbb{Z}$. If $n^3$ is a multiple of 3, then $n$ is odd.

**The method:** 6) ✓

$\frac{5}{5}$

Hyp. If $n^3$ is a multiple of 3, $n$ is odd.

For $n^3 = 256$ $(256 | 3)$

$n = 6$ $6 \in \mathbb{Z}$ and not odd ✓

$= 2(k)$ for $k \in \mathbb{Z}$ and $k = 3$

$n$ is even and not odd

$\therefore$ The statement is false disproved by counterexample.

[*5 marks*] (b) Let $a$ and $b$ be real numbers with $a < b$. There exist a real number $c$ such that $a < c < b$.

**The method:** 2) ok $p$ $q$

$\frac{1}{5}$

For $a, b, c \in R$

$\sim q \Rightarrow$ there does not exist a real number $c$ such that $a < c < b$, $a, b, c \in \mathbb{R}$

$\Rightarrow a = b$ since $c$ doesn't exist

$\Rightarrow (a = b) \neq (a < b) \Rightarrow \sim p$

$\sim q \rightarrow \sim p$

$-4$

$\therefore$ The statement is proved by contrapositive (TRUE)

[*5 marks*] (c) Let $a$ and $b$ be real numbers. If $|a + b| < |a| + |b|$, then $a \neq b$.

**The method:** 2) X $p$ $q$

$\frac{4}{5}$

If $a = b$ $(\sim q)$ $q$ is false

then $|a+b| < |a| + |b|$ for $a, b \in \mathbb{R}$

$\Rightarrow |b+b| < |b| + |b|$

$\Rightarrow |2b| < 2|b|$

$\Rightarrow 2|b| < 2|b|$ since $2 > 0, 2 \in \mathbb{R}$

$\Rightarrow |b| < |b|$

$\Rightarrow |b| = |b| \neq \underbrace{|b| < |b|}_{\text{doesn't exist}}$

ok.

$\therefore$ The statement is True proved by contrapositive

[*5 marks*] (d) Let Let $A$ and $B$ be sets. If $A \cap B = A \cup B$, then $A = B$.

4) *Non-constructive proof of existence*

5) *Proof of existence by construction*

6) *Disproof by counterexample*

[5 marks] (a) Let $n \in \mathbb{Z}$. If $n^3$ is a multiple of 3, then $n$ is odd.

**The method:** 6) ✓

Hyp. If $n^3$ is a multiple of 3, $n$ is odd.

5/5

For $n^3 = 256$ $(256|3)$

$n = 6$ $6 \in \mathbb{Z}$ and not odd ✓

$= 2(k)$ for $k \in \mathbb{Z}$ and $k = 3$

$n$ is even and not odd

$\therefore$ The statement is false disproved by counterexample.

[5 marks] (b) Let $a$ and $b$ be real numbers with $a < b$. There exist a real number $c$ such that $a < c < b$.

**The method:** 2) ok $p$ $q$

1/5

For $a, b, c \in R$

$\sim q \Rightarrow$ there does not exist a real number $c$ such that $a < c < b$, $a, b, c \in \mathbb{R}$

$\Rightarrow a = b$ since $c$ doesn't exist

$\Rightarrow (a = b) \neq (a < b) \Rightarrow \sim p$

$\sim q \to \sim p$

$-4$

$\therefore$ The statement is proved by contrapositive (TRUE)

[5 marks] (c) Let $a$ and $b$ be real numbers. If $|a + b| < |a| + |b|$, then $a \neq b$.

**The method:** 2) ✗ $p$ $q$

4/5

If $a = b$ $(\sim q)$ $q$ is false

then $|a+b| < |a| + |b|$ for $a, b \in \mathbb{R}$

$\Rightarrow |b+b| < |b| + |b|$

$\Rightarrow |2b| < 2|b|$

$\Rightarrow 2|b| < 2|b|$ since $2 > 0$, $2 \in \mathbb{R}$

$\Rightarrow |b| < |b|$

$\Rightarrow |b| = |b| \neq \underbrace{|b| < |b|}_{\text{doesn't exist}}$

ok.

$\therefore$ The statement is True proved by contrapositive

[5 marks] (d) Let Let $A$ and $B$ be sets. If $A \cap B = A \cup B$, then $A = B$.

**The method:** 1) ok $p$ $q$

1/5

For A and B are sets

$A = A \cup A = A \cap A$

$B = B \cup B = B \cap B$

$A \cap B = A \cup B$

$\Rightarrow A \cap (A) = A \cup (A)$

or

$(B) \cap B = (B) \cup (B)$

$\Rightarrow A \cap B = A \cup B$

$\Rightarrow$ B must equal A to be true

Show nothing $-4$

4) *Non-constructive proof of existence*

5) *Proof of existence by construction*

6) *Disproof by counterexample*

[*5 marks*] (a) Let $n \in \mathbb{Z}$. If $n^3$ is a multiple of 3, then $n$ is odd.

The method: 6) ✓

Hyp. If $n^3$ is a multiple of 3, n is odd.

5/5

For $n^3 = 256$ $(256|3)$

$n = 6$ $6 \in \mathbb{Z}$ and not odd ✓

$= 2(k)$ for $k \in \mathbb{Z}$ and $k = 3$

n is even and not odd

∴ The statement is false disproved by counterexample.

[*5 marks*] (b) Let $a$ and $b$ be real numbers with $a < b$. There exist a real number $c$ such that $a < c < b$.

The method: 2) ok

$p$ $q$

For $a, b, c \in R$

1/5

$\sim q \Rightarrow$ there does not exist a real number $c$ such that $a < c < b$, $a, b, c \in \mathbb{R}$

$\Rightarrow a = b$ since c doesn't exist

$\Rightarrow (a = b) \neq (a < b) \Rightarrow \sim p$

$\sim q \to \sim p$

−4

∴ The statement is proved by contrapositive (TRUE)

[*5 marks*] (c) Let $a$ and $b$ be real numbers. If $|a + b| < |a| + |b|$, then $a \neq b$.

The method: 2) ✗

$p$ $q$

If $a = b$ $(\sim q)$ q is false

4/5

then $|a+b| < |a| + |b|$ for $a, b \in \mathbb{R}$

$\Rightarrow |b+b| < |b| + |b|$

$\Rightarrow |2b| < 2|b|$

$\Rightarrow 2|b| < 2|b|$ since $2 > 0, 2 \in \mathbb{R}$

$\Rightarrow |b| < |b|$

$\Rightarrow |b| = |b| \neq \underbrace{|b| < |b|}_{\text{doesn't exist}}$

ok.

∴ The statement is True proved by contrapositive

[*5 marks*] (d) Let Let $A$ and $B$ be sets. If $A \cap B = A \cup B$, then $A = B$.

The method: 1) ok

$p$ $q$

For A and B are sets

1/5

$A = A \cup A = A \cap A$

$B = B \cup B = B \cap B$

$A \cap B = A \cup B$

$\Rightarrow A \cap (A) = A \cup (A)$

or

$(B) \cap B = (B) \cup (B)$

$\Rightarrow A \cap B = A \cup B$

Show nothing

−4

$\Rightarrow$ B must equal A to be true

[*6 marks*] 7. Let $U = \{a, b, c, d, e\}$ be the universal set. Let $A = \{a, b, c\}$, $B = \{b, c, d\}$ and $C = \{c, d, e\}$. Determine:

$\frac{6}{7}$

(a) $(A \cup B) \cap C =$ $\{c, d\}$ ✓

(b) $(A \cap B) \cup C =$ $\{b, c, d, e\}$ ✓

(c) $A \cap B \cap C =$ $\{c\}$ ✓

(d) $A \cup B \cup C$ ~~$A \cup B \cap C$~~ $=$ $\{a, b, c, d, e\}$ ✓

(e) $U \cap \emptyset \cap \{\emptyset\} =$ $\emptyset$ ✓

(f) $U \cup \emptyset \cup \{\emptyset\} =$ $U$ ✗

(g) $(A \cap U) \cup B =$ $\{a, b, c, d\}$ ✓

[*9 marks*] 8. (a) Prove in the **language of set theory** that for any sets $A$, $B$ and $C$,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

$\frac{9}{9}$

For $x \in A \cap (B \cup C)$

$\Rightarrow x \in A$ and $x \in (B \cup C)$

$\Rightarrow x \in A$ and $x \in (B$ or $C)$

$\Rightarrow x \in A$ and $(x \in B$ or $x \in C)$

$\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Rightarrow x \in (A \cap B)$ or $x \in (A \cap C)$

$\Rightarrow (A \cap B) \cup (A \cap C)$

$\Rightarrow A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

For $x \in [(A \cap B) \cup (A \cap C)]$

$\Rightarrow x \in (A$ or $B)$ and $(A$ or $C)$

$\Rightarrow [x \in A$ or $x \in B]$ and $[x \in A$ or $x \in C]$

$\Rightarrow x \in A$ and $(x \in B$ or $x \in C)$

$\Rightarrow x \in A \cap (x \in B \cup C)$

$\Rightarrow x \in A \cap (B \cup C)$

$\Rightarrow A \cap (B \cup C)$

$\Rightarrow (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

Since $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ ✓

$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(c) $A\cap B\cap C =$ $\{c\}$ ✓

7

(d) $A\cup B\cup C$ ~~$A\cup B\cap C$~~ $= \{a,b,c,d,e\}$

(e) $U\cap\emptyset\cap\{\emptyset\} =$ $\emptyset$ ✓

(f) $U\cup\emptyset\cup\{\emptyset\} =$ $U$ ✗

(g) $(A\cap U)\cup B =$ $\{a,b,c,d\}$ ✓

[*9 marks*] 8. (a) Prove in the **language of set theory** that for any sets $A$, $B$ and $C$,

$$A\cap(B\cup C) = (A\cap B)\cup(A\cap C).$$

9/9

For $x\in A\cap(B\cup C)$

$\Rightarrow x\in A$ and $x\in(B\cup C)$

$\Rightarrow x\in A$ and $x\in(B$ or $C)$

$\Rightarrow x\in A$ and $(x\in B$ or $x\in C)$

$\Rightarrow (x\in A$ and $x\in B)$ or $(x\in A$ and $x\in C)$

$\Rightarrow x\in(A\cap B)$ or $x\in(A\cap C)$

$\Rightarrow (A\cap B)\cup(A\cap C)$

$\Rightarrow A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$

For $x\in[(A\cap B)\cup(A\cap C)]$

$\Rightarrow x\in(A$ or $B)$ and $(A$ or $C)$

$\Rightarrow [x\in A$ or $x\in B]$ and $[x\in A$ or $x\in C]$

$\Rightarrow x\in A$ and $(x\in B$ or $x\in C)$

$\Rightarrow x\in A\cap(x\in B\cup C)$

$\Rightarrow x\in A\cap(B\cup C)$

$\Rightarrow A\cap(B\cup C)$

$\Rightarrow (A\cap B)\cup(A\cap C)\subseteq A\cap(B\cup C)$

Since $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap$

and $(A\cap B)\cup(A\cap C)\subseteq A\cap(B\cup C)$

✓

$\therefore A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

▨

[*3 marks*] (b) Given simple statements

3/3

$$p : x\in A$$
$$q : x\in B$$
$$r : x\in C$$

Write down the tautology in propositional logic underlying the proof in part (a) in terms of simple statements $p$, $q$, $r$ and connectives $\lor$, $\land$, $\leftrightarrow$.

$A\cap(B\cup C)$

↓ ↓ ↓

p q r $\quad p\land(q\lor r) \Longleftrightarrow (p\land q)\lor(p\land r)$