Instructions:

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Answer in the spaces provided.

Show all your work. Insufficient justification will result in a loss of marks.

1. [8 *marks*] Find an equation of the curve $y = f(x)$ that satisfies the differential equation $\frac{dy}{dx} = y \sin^{-1} x$ and passes through the point $(1, 1)$. **[Hint:** Use integration by parts.]

$$\frac{dy}{dx} = y \sin^{-1} x$$

$$\int \frac{1}{y}\, dy = \int \sin^{-1} x \, dx$$

$$\ln|y| = x \sin^{-1} x + \sqrt{1-x^2} + C$$

$$|y| = e^{x\sin^{-1}x + \sqrt{1-x^2} + C}$$

so $y = \pm e$

For (1,1)

$$1 = e^{1 \sin^{-1}(1) + \sqrt{1-1^2} + C}$$

$$\ln 1 = \frac{\pi}{2} + 0 + C$$

$$0 = \frac{\pi}{2} + C$$

$$C = \frac{-\pi}{2}$$

$$y = f(x)$$

$$\Downarrow$$

$$y = e^{x\sin^{-1}x + \sqrt{1-x^2} - \frac{\pi}{2}}$$

$u = \sin^{-1} x$ $\quad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}} dx$ $\quad v = x$

$uv - \int v\, du$

$$\int v \, du$$

$$= \int x \left(\frac{1}{\sqrt{1-x^2}}\right) dx$$

$$= \frac{-1}{2} \int \frac{1}{\sqrt{t}}\, du$$

let $t = 1 - x^2$

$$= \frac{-1}{2} \int 2t^{\frac{1}{2}} + C$$

$dt = 2x\,dx$

$$= -t^{\frac{1}{2}} + C$$

$\frac{-1}{2} dt = x\,dx$

$t = 1 - x^2$

$-t^{\frac{1}{2}} = -\sqrt{1-x^2}$

series $\neq$ sequence

7/8

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x$$

2. Evaluate the following indefinite integrals:

(a) [5 *marks*] $\int \tan^5(x)\sec^3(x)\ dx$

$(\tan^4 x \quad \sec^2 x) \cdot \underline{\sec x \cdot \tan x}$

$(\sec^2 - 1)^2$

$\sin^2 + \cos^2 = 1$

$\tan^2 + 1 = \sec^2$

5

5/5

$$\int \tan^5 x \sec^3 x\, dx$$

$$= \int \tan^4 x \tan x \cdot \sec^3 x\, dx$$

$$= \int (\sec^2 x - 1)^2 \cdot \sec^2 x \cdot (\tan x \sec x)\, dx \quad \text{let } u = \sec x, \; du = \tan x \sec x\, dx$$

$$= \int (u^2-1)^2 \cdot u^2\, du$$

$$= \int (u^4 - 2u^2 + 1)u^2\, du$$

$$= \int u^6 - 2u^4 + u^2\, du$$

$$= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C$$

$$\boxed{= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C}$$

(b) [5 *marks*] $\int \sin^4 x\ dx$

$$= \int (\sin^2 x)^2\, dx$$

$$= \int \left(\tfrac{1}{2}(1 - \cos 2x)\right)^2 dx$$

$$= \frac{1}{4}\int (1 - \cos 2x)^2\, dx$$

$$= \frac{1}{4}\int 1 - 2\cos 2x + \cos^2 2x\, dx$$

$$= \frac{1}{4}\int 1 - 2\cos 2x + \tfrac{1}{2}(1 + \cos 4x)\, dx$$

$$= \frac{1}{4}\int \tfrac{3}{2} - 2\cos 2x + \tfrac{1}{2}\cos 4x\, dx$$

$$= \frac{1}{4}\left[\frac{3x}{2} - \sin 2x + \frac{1}{8}\sin x\right] + C$$

(4x)

4/5

$$\boxed{= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin x}{32} + C}$$

4x

(c) [5 *marks*] $\displaystyle\int \frac{dx}{x^2\sqrt{16 - x^2}}$

$$= \int \frac{1}{x^2\sqrt{16 - x^2}}\, dx \quad \text{let } x = 4\sin\theta, \; -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \; dx = 4\cos\theta\, d\theta$$

$$= \int \frac{1}{(16\sin^2\theta)\sqrt{16 - 16\sin^2\theta}}(4\cos\theta\, d\theta)$$

$$= \int \frac{4\cos\theta}{16\sin^2\theta\sqrt{16(1-\sin^2\theta)}}\, d\theta$$

$$= \int \frac{4\cos\theta}{16\sin^2\theta \cdot (4\cos\theta)}\, d\theta$$

4/5

$$= \frac{1}{16}\int \frac{1}{\sin^2\theta}\, d\theta$$

3.

(a) [6 *marks*] Evaluate the improper integral $\int_0^1 \frac{e^x}{e^x - 1}\, dx$ or show that it is divergent.

For $y = f(x)$

$= \frac{e^x}{e^x - 1}$

$f(1) = \frac{e}{e-1}$

$f(0) = \frac{1}{1-1} = \frac{1}{0} = \text{DNE}$

$a \le c \le b$ ↑ cont.

$\int_a^{c^-} + \int_{c^+}^{b} dx$

$\lim_{t \to 0^+} \int_t^1 \frac{e^x}{e^x - 1}\, dx$

let $u = e^x - 1$
$du = e^x dx$

$= \lim_{t \to 0^+} \int_{x=t}^{x=1} \frac{1}{u}\, du$

6/6

$= \lim_{t \to 0^+} \ln|u| \Big]_{x=t}^{x=1}$

$= \lim_{t \to 0^+} \ln|e^x - 1| \Big]_t^1$

$= \lim_{t \to 0^+} \left(\ln|e^1 - 1| - \underbrace{\ln|e^t - 1|}_{\to -\infty} \right) = \ln|e-1| + \infty$

$= \infty$ ∴ it is divergent.

(b) [5 *marks*] Use the Comparison Theorem to determine whether the improper integral below is convergent or divergent:

$$\int_1^\infty \frac{2 + e^{-x}}{x}\, dx.$$

[**Hint:** $e^{-x} > 0$ for all x.]

since the integral is from $(1, \infty)$, x must be a positive value.

$\frac{2 + e^{-x}}{x} \le 2 + e^{-x}$ for $x \in [1, \infty)$

↳ show convergence

1/5

$\int_1^\infty 2 + e^{-x} dx$

$= \lim_{t \to \infty} \int_1^t 2 + e^{-x} dx$

$= \lim_{t \to \infty} 2x + e^{-x} \Big]_1^t$

$= \lim_{t \to \infty} (2t - e^{-t} - 2 + e^{-1})$

$\int \frac{2 + e^{-x}}{e^{x}}\, dx$?

$\int \frac{2}{e^x} + \frac{e^{-x}}{e^x}\, dx$

$= \int 2e^{-x} + e^{-2x}\, dx$

$= -2e^{-x} + \frac{1}{2}e^{-2x}$

4. Let C be the curve $y = \frac{x^2}{2} - \frac{\ln x}{4}$, $2 \le x \le 4$.

(a) [6 *marks*] Find the length of C.

$$L = \int \sqrt{1+(f'(x))^2}\, dx$$

$$y = \frac{x^2}{2} - \frac{\ln x}{4}$$

$$y' = x - \frac{1}{4x}$$

$$2x \cdot \frac{\frac{1}{4x}}{2}$$

$$L = \int_2^4 \sqrt{1+(x-\tfrac{1}{4x})^2}\, dx$$

$$= \int_2^4 \sqrt{1+x^2-\tfrac{1}{2}+\tfrac{1}{16x^2}}\, dx$$

$$= \int_2^4 \sqrt{\tfrac{1}{2}+x^2+\tfrac{1}{16x^2}}\, dx$$

$$= \int_2^4 \sqrt{\frac{8x^2+16x^4+1}{16x^2}}\, dx$$

6/6

$$= \int_2^4 \frac{1}{4x}\sqrt{16x^4+8x^2+1}\, dx$$

$$= \int_2^4 \frac{1}{4x}\sqrt{(4x^2+1)^2}\, dx$$

$$= \int_2^4 \frac{4x^2+1}{4x}\, dx$$

$$= \int_2^4 x + \frac{1}{4x}\, dx$$

$$= \frac{1}{2}x^2 + \frac{1}{4}\ln|x| \Big]_2^4$$

$$= \frac{16}{2} + \frac{\ln 4}{4} - \frac{4}{2} - \frac{\ln 2}{4}$$

$$\boxed{\approx 6.1733}$$

(b) [4 *marks*] Find the area of the surface formed by rotating C about the y-axis.

$$SA = \int 2\pi x \sqrt{1+(f'(x))^2}\, dx$$

$$SA = \int_2^4 2\pi x \sqrt{1+(x-\tfrac{1}{4x})^2}\, dx$$

$$= \int_2^4 2\pi x\left(\frac{4x^2+1}{4x}\right) dx$$

$$= \frac{\pi}{2}\int_2^4 4x^2+1\, dx$$

3/4

$$= \frac{\pi}{2}\left[\frac{4}{3}x^3+x\right]_2^4$$

$$= \frac{\pi}{2}\left[\left(\frac{256}{3}+4\right)-\left(\frac{32}{3}+2\right)\right]$$

careful! calculation error

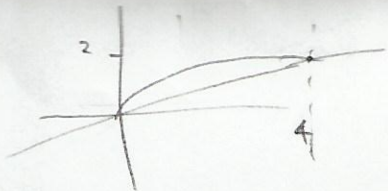
$$\boxed{\approx 126.7109}$$

x

≈ 120.428

5. [7 *marks*] Let $\mathcal{R}$ denote the region bounded between the curves $y = \frac{x}{2}$ and $y = \sqrt{x}$. Find the volume of the solid obtained by rotating $\mathcal{R}$ about the line $x = 4$.

$x = 2y \qquad x = y^2$



$$A = \int_0^2 \pi \left[(4-y^2)^2 - (4-2y)^2 \right] dy$$

$$= \pi \int_0^2 16 - 8y^2 + y^4 - \overset{16}{4} + 16y - 4y^2 \, dy$$

$$= \pi \int_0^2 y^4 - 12y^2 + 16y + \cancel{12} \, dy$$

6/7

$$\frac{1}{5}y^5 - 4y^3 + 8y^2$$

$$\frac{32}{5} - 32 + 32$$

$$= \pi \left[\frac{1}{5}y^5 - 4y^3 + 8y^2 + 12y \right]_0^2$$

$$= \frac{152}{5}\pi$$

$$\boxed{\doteq 95.5044}$$

6. [6 *marks*] A circle with radius r and centred at $(0, 0)$ is given parametrically by

$$x = r\cos t, \quad y = r\sin t, \quad 0 \le t \le 2\pi.$$

Evaluate an appropriate integral to verify that the surface area of a sphere is given by $4\pi r^2$.

$x = r\cos t$

$y = r\sin t \qquad 0 \le t \le 2\pi$

$\frac{dx}{dt} = -r\sin t$

$\frac{dy}{dt} = r\cos t$

$\frac{4}{3}\pi r^3$

$\frac{dy}{dx} = \frac{r\cos t}{-r\sin t}$

$= -\cot$

$$SA = \int_0^{2\pi} \overset{y}{2r} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

2/6

would only rotate half a circle to get a sphere.

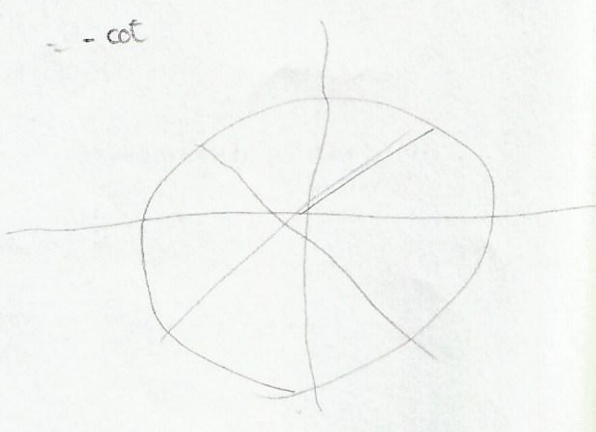
$$= \int_0^{2\pi} 2r \sqrt{r^2\cos^2 t + r^2\sin^2 t} \, dt$$

$$= \int_0^{2\pi} 2r \sqrt{r^2(\cos^2 t + \sin^2 t)} \, dt$$

$$\int_0^{2\pi} 2r^2 \, dt$$

$$= 2r^2 (t) \Big]_0^{2\pi}$$

$$= 2r^2 (2\pi - 0)$$



$x = \frac{t}{2} + \cos t \qquad 0 \le t \le \pi \qquad \frac{dx}{dt} = \frac{1}{2} - \sin t$ ✓

$y = \frac{t}{2} + \sin t \qquad \frac{dy}{dt} = \frac{1}{2} + \cos t$

$L = \int_0^{\pi} \cancel{1+} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$

2/4

$= \int_0^{\pi} \sqrt{\left(\frac{1}{2} - \sin t\right)^2 + \left(\frac{1}{2} + \cos t\right)^2}\, dt$

$= \int_0^{\pi} \sqrt{\frac{1}{4} - \sin t + \sin^2 t + \frac{1}{4} + \cos t + \cos^2 t}\, dt$

$= \int_0^{\pi} \sqrt{\frac{1}{2} + \cos t - \sin t + 1}\, dt$

(b) [6 *marks*] Approximate the arc length from part(a) by using the Trapezoidal Rule and $n = 4$ equal subintervals. ✓

$n = 4$

$0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

$f(t) =$

$\Delta t = \frac{\pi - 0}{4} = \frac{\pi}{4}$

$t_4: 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

$f(t) = \sqrt{\frac{3}{2} + \cos t - \sin t}$

where $f(t)$ is ... (INCLUDE)

$L \approx T_4 = \frac{\Delta t}{2}\left[f(t_0) + 2f(t_1) + 2f(t_2) + 2f(t_3) + f(t_4)\right]$

$= \frac{\pi}{8}\left(\ldots \right)$

$=$

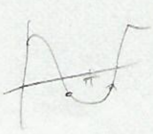
$L \approx 2.646$

(c) [4 *marks*] Determine the (x, y) points, if any, at which the tangent line to the curve segment is vertical.

$\frac{dy}{dt} \ne 0 \qquad \frac{dx}{dt} = 0$

$\frac{dy}{dt} = \frac{1}{2} + \cos t \qquad \frac{dx}{dt} = \frac{1}{2} - \sin t$

$\cos t = -\frac{1}{2} \qquad \sin t = -\frac{1}{2}$



$t = \frac{2\pi}{3}, \frac{4\pi}{3} \qquad t = \frac{\pi}{3}, \frac{\pi}{6}$

$t =$

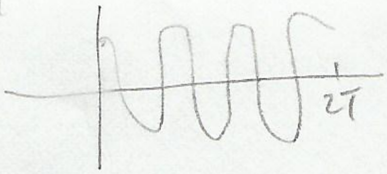
$x = \frac{\pi}{6} + \cos\frac{\pi}{3} \qquad y = \frac{\pi}{6} + \sin\frac{\pi}{6}$

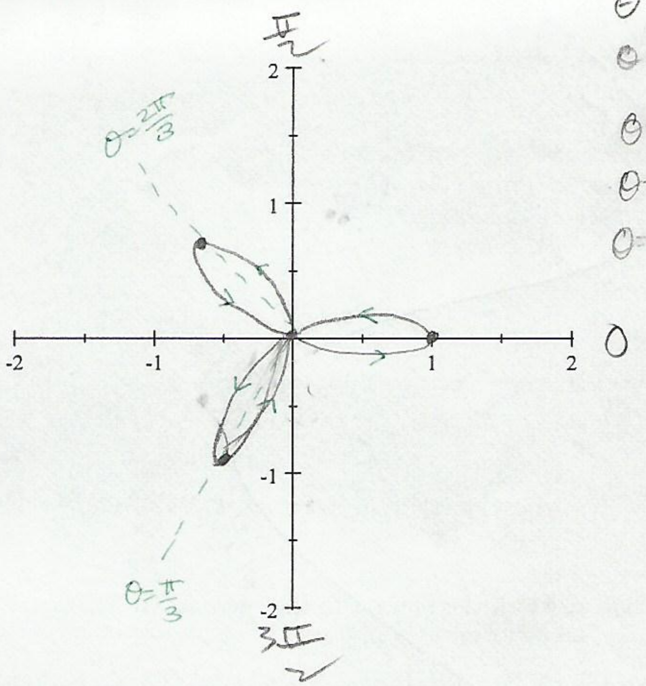
$(\ , \ ) \quad (\ , \ )$

8. [11 marks]

(a) Sketch the 3-leafed rose given by the polar curve $r = \cos(3\theta)$. $+1$

$y = \cos 3x$





$\theta = 0 \quad r = 1$

$\theta = \frac{\pi}{3} \quad r = -1$

$\theta = \frac{\pi}{6} \quad r = 0$

$\theta = \frac{\pi}{4} \quad r = \frac{-\sqrt{2}}{2}$

$\theta = \pi \quad r = -1$

3/3

$2l^2 = 3$

$l = \sqrt{\frac{3}{2}}$

$\sqrt{\frac{3}{2}}^2 + \sqrt{\frac{3}{2}}^2$

$= \frac{3}{2} + \frac{3}{2}$

$= 3$

(b) Determine the area enclosed by one leaf of the rose.

$$\int_0^{\frac{2\pi}{3}} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

$$\cancel{= \int_0^{\frac{2\pi}{3}} \sqrt{(-3\sin 3\theta)^2 + (\cos 3\theta)^2}\, d\theta}$$

$$\cancel{= \int_0^{\frac{2\pi}{3}} \sqrt{9\sin^2 3\theta + \cos^2 3\theta}\, d\theta}$$

$r = \cos 3\theta$

$\frac{dr}{d\theta} = -3\sin 3\theta$

$$A = \int \frac{1}{2}(f(\theta))^2\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2} \cos^2 3\theta\, d\theta$$

$$= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 6\theta)\, d\theta$$

$$= \frac{1}{4} \left[\theta + \frac{\sin 6\theta}{6}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$\frac{\pi}{6}$

$$= \frac{\frac{\pi}{3} + 0}{4} = \frac{\pi}{12}$$

(c) Find the slope of the tangent to r at $\theta = \frac{\pi}{4}$.

$r = \cos 3\theta$

$y = r\cos\theta$

$x = r\sin\theta$

$\frac{dr}{d\theta} = -3\sin 3\theta$

$1/1$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$= \frac{-3\sin 3\theta \sin\theta + \cos 3\theta \cos\theta}{-3\sin 3\theta \cos\theta - \cos 3\theta \sin\theta} = \frac{-3\sin\frac{3\pi}{4}\sin\frac{\pi}{4} + \cos\frac{3\pi}{4}\cos\frac{\pi}{4}}{-3\sin\frac{3\pi}{4}\cos\frac{\pi}{4} - \cos\frac{3\pi}{4}\sin\frac{\pi}{4}}$$

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$\frac{dy}{dx} = y\sin^{-1}x$

$\int \frac{1}{y}\,dy = \int \sin^{-1}x\,dx$ ✓

✓ $\ln|y| = x\sin^{-1}x + \sqrt{1-x^2} + C$ ✓

$|y| = e^{x\sin^{-1}x + \sqrt{1-x^2} + C}$

so $y = \pm e$

7/8

For $(1,1)$

$1 = e^{1\sin^{-1}(1) + \sqrt{1-1^2} + C}$

$\ln 1 = \frac{\pi}{2} + 0 + C$

$0 = \frac{\pi}{2} + C$ ✓

$C = \frac{-\pi}{2}$

$y = f(x)$

$\Downarrow$

$y = e^{x\sin^{-1}x + \sqrt{1-x^2} - \frac{\pi}{2}}$ ✓

$u = \sin^{-1}x \qquad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}}\,dx \qquad v = x$

$uv - \int v\,du$

$\int v\,du$

$= \int x\left(\frac{1}{\sqrt{1-x^2}}\right)dx$

$= \frac{-1}{2}\int \frac{1}{\sqrt{t}}\,dt \qquad$ let $t = 1-x^2$

$= \frac{-1}{2} 2t^{\frac{1}{2}} + C \qquad dt = -2x\,dx$

$= -t^{\frac{1}{2}} + C \qquad \frac{-1}{2}dt = x\,dx$

$t = 1-x^2$ ✓

$-t^{\frac{1}{2}} = -\sqrt{1-x^2}$

series
sequence $\neq$