the course.

Time Allowed: 150 minutes
Total Value: 100 marks
Number of Pages: 9

Instructions:

Cheat Sheet: One 8.5" × 11" page of study notes (both sides) is allowed as a reference while completing the mock test. Please note, that the cheat sheet is permitted for the mock test only!!

Non-programmable, non-graphing calculators are permitted. No other aids allowed.

Check that your test paper has no missing, blank, or illegible pages. Note that test questions appear on ***both*** *sides of the paper.*

Answer in the spaces provided.

Show all your work. Insufficient justification will result in a loss of marks.

1. [7 *marks*] Consider the two-variable function $z = f(x, y) = \sqrt{20 - 2x^2 - y^2}$.

(a) Determine an equation of the plane tangent to f at $(1, 3)$.

$f(x,y) = \sqrt{20-2x^2-y^2}$

$f(1,3) = 3$

$f_x(x,y) = \frac{1}{2}(20-2x^2-y^2)^{-\frac{1}{2}} \cdot (-4x)$ ✓

$f_y(x,y) = \frac{1}{2}(20-2x^2-y^2)^{-\frac{1}{2}} \cdot (-2y)$

4/4

$f_x(1,3) = \frac{1}{2}(20-2-9)^{-\frac{1}{2}}(-4)$ ✓

$= \frac{-2}{3}$

$$\boxed{z = \frac{-2}{3}x - y + \frac{20}{3}}$$

$f_y(1,3) = -1$

$z - 3 = \frac{-2}{3}(x-1) + (-1)(y-3)$ ✓

$z = \frac{-2}{3}x + \frac{2}{3} - y + 3 + 3$

(b) Use a tangent plane approximation (i.e., a linear approximation) to estimate the value of $f(1.01, 2.97)$

$L(x,y) = \frac{-2}{3}x - y + \frac{20}{3}$

2/3

$f(1.01, 2.97) \approx L(1.01, 2.97) = \frac{-2}{3}(1.01) - 2.97 + \frac{20}{3}$

$= 3.02\overline{3}$

Real value = 3.023

$$\boxed{= \frac{907}{300}}$$ ✓

make sure you connect your answer to what was asked.

2. [~~10~~ 12 *marks*] Consider the two variable function $f(x,y) = 3x - x^3 - 3xy^2$.

(a) Find all critical points and determine the relative extrema (if any) of f. Use the Second Derivative Test to justify your conclusions.

$f(x,y) = 3x - x^3 - 3xy^2$

$f_x(x,y) = 3 - 3x^2 - 3y^2$

$f_y(x,y) = -6xy$ ✓

$f_x = 0$

$f_y = 0$ when $x=0$, $y = 1$ or -1

or

$y = 0$, $x = 1$ or -1

$(0,1), (0,-1), (1,0), (-1,0)$ ✓

6/6

$f_{xx}(x,y) = -6x$

$f_{xy}(x,y) = -6y$

$f_{yy}(x,y) = -6x$ ✓

$D = f_{xx}f_{yy} - (f_{xy})^2$

$= 36x^2 - 36y^2$

Pts

$(0,1) \to D = 0 - (-6)^2 = -36$

$\Rightarrow$ saddle point $\qquad f(0,1) = 0$

$(0,-1) \to D = 0 - (6)^2 = -36$ ✓

$\Rightarrow$ saddle point $\qquad f(0,-1) = 0$

$(1,0) \to D = (-6)(-6) - 0 = 36$

$f_{xx} = -6$

$\Rightarrow$ relative Maximum point $\qquad f(1,0) = 2$

$(-1,0) \to D = 6 \cdot 6 - 0 = 36$ ✓

$f_{xx} = 6$

$\Rightarrow$ relative minimum point $\qquad f(-1,0) = -2$

$\therefore$ The critical points are $(0,1,0), (0,1,0), (1,0,2)$ and $(-1,0,-2)$ where $(0,1,0)$ and $(0,-1,0)$ are saddle points and $(1,0,2)$ is a local maximum and $(-1,0,-2)$ is a local minimum. ✓

(b) Find the absolute maximum and minimum values of f on the set D where D is the closed triangular region with vertices $(3,0)$, $(0,3)$, $(0,-3)$.

State Domain + Range

$f(1,0) = 2$ (local max.)

$L_1 \Rightarrow y = 3 - x$

$L_2 \Rightarrow y = x - 3$

$L_3 \Rightarrow x = 0$

4/6

L_1

$f(x,y) = f(x, 3-x)$, $0 \le x \le 3$

$= 3x - x^3 - 3x(3-x)^2$

$f(x,y) = 3x - x^3 - 3xy^2$

$(0, 3, ?)$ 0

L_1: Critical points $(1, 2, \cancel{-36})$ -10

$(2, 1, -8)$

$(3, 0, ?)$ -18

need to check endpoints of L_1 as well as critical pts along L_1.

L_2: Critical points $(1, -2, -10)$ -18

$(2, -1, -8)$

$(0, -3, ?)$ 0

same here

$L_3 := 0$

$(3, 0, \cancel{18}\ \cancel{-18})$

$f(1,0) = 2$

$f_{xx}(x,y) = -6x$
$f_{xy}(x,y) = -6y$
$f_{yy}(x,y) = -6x$ ✓

$D = f_{xx}f_{yy} - (f_{xy})^2$
$= 36x^2 - 36y^2$

$(-1,0) \to D = 6 \cdot 6 - 0 = 36$ ✓
$f_{xx} = 6$
relative ⇒ minimum point

$f(-1,0) = -2$

∴ The critical points are (0,1,0), (0,1,0), (1,0,2) and (-1,0,-2) where (0,1,0) and (0,-1,0) are saddle points and (1,0,2) is a local maximum and (-1,0,2) is a local minimum. ✓

(b) Find the absolute maximum and minimum values of f on the set D where D is the closed triangular region with vertices $(3,0)$, $(0,3)$, $(0,-3)$.

State Domain + Range

4/6

$f(x,y) = 3x - x^3 - 3xy^2$

$f(1,0) = 2$ (local max.)

$L_1 \Rightarrow y = 3 - x$
$L_2 \Rightarrow y = x - 3$
$L_3 \Rightarrow x = 0$

L_1: Critical points $(0, 3, 0)$, $(1, 2, \cancel{-130})$ -10, $(2, 1, -8)$

need to check endpoints of L_1 as well as critical pts along L_1.

L_2: Critical points $(3, 0, -18)$, $(1, -2, -10)$, $(2, -1, -8)$, $(0, -3, ?)$ 0

same here

$L_3 := 0$

$(3, 0, -18)$

$f(1,0) = 2$

L_1
$f(x,y) = f(x, 3-x)$, $0 \le x \le 3$

$= 3x - x^3 - 3x(3-x)^2$
$= 3x - x^3 - 3x(9 - 6x + x^2)$
$= 3x - x^3 - 27x + 18x^2 - 3x^3$
$= -4x^3 + 18x^2 - 24x$ ✓

$\frac{\partial L_1}{\partial x} = -12x^2 + 36x - 24 = 0$
$= x^2 - 3x + 2 = 0$
$x = 1, 2$ $\quad x = 1, 2$
$y = 2, 1$

$x = -2$ would not be part of D.

Fuck

Absolute Max $= (1, 0, 2)$ ✓
Absolute Min $= \cancel{(1, 2, -130)}$ ✗

$(3, 0, -18)$

~~$(-3, 0, -18)$~~

L_2
$f(x, x-3) = 3x - x^3 - 3x(x-3)^2$, $0 \le x \le 3$
$=$
$\vdots$
$= -4x^3 + 18x^2 - 24x$ ✓

$\frac{\partial L_2}{\partial x} = 0$

$x = 1, 2$
$y = -2, -1$ ok!

L_3
$f(0,y) = 0$ ✓
$-3 \le y \le 3$

3. [7 *marks*] Determine the area of the surface generated by rotating the curve $x = e^{4t} - t$, $y = 2e^{2t}$, $0 \le t \le 2$ about the x-axis.

$\frac{dx}{dt} = 4e^{4t} - 1$

$\frac{dy}{dt} = 4e^{2t}$

$$SA = 2\pi \int_0^2 2e^{2t}\sqrt{(4e^{4t}-1)^2 + (4e^{2t})^2}\, dt$$

$$= \int_0^2 2e^{2t}\sqrt{16e^{8t} - 8e^{4t} + 1 + 16e^{4t}}\, dt$$

$$= \int_0^2 2e^{2t}\sqrt{16e^{8t} + 8e^{4t} + 1}\, dt$$

$$= \int_0^2 2e^{2t}\sqrt{(4e^{4t}+1)^2}\, dt$$

$$= \int_0^2 2e^{2t}(4e^{4t}+1)\, dt$$

$$= \int_0^2 8e^{6t} + 2e^{2t}\, dt$$

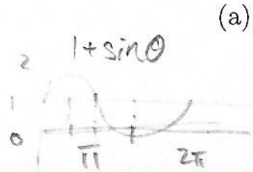
$$= \frac{4}{3}e^{6t} + e^{2t}\Big]_0^2$$

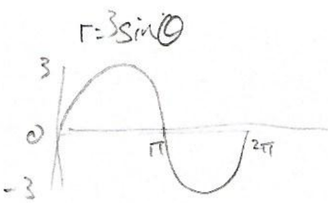
$$= \frac{4}{3}e^{12} + e^{4} - \frac{4}{3} - 1 = \boxed{217{,}058.65}$$

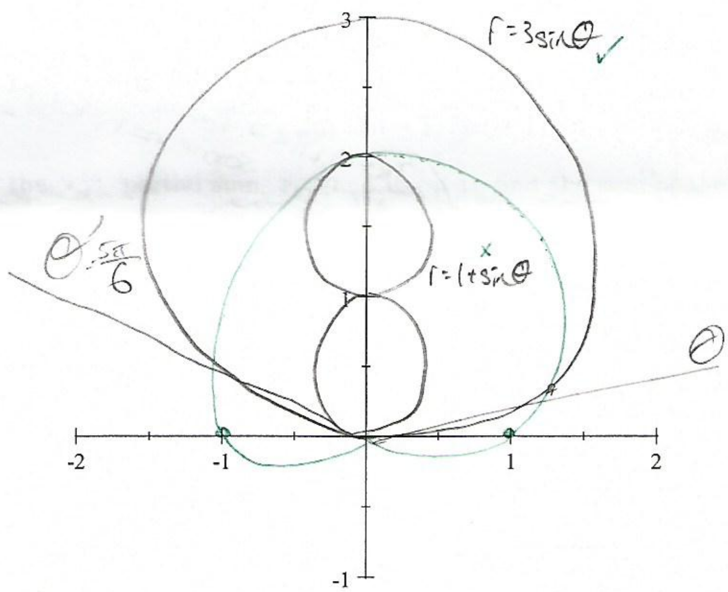
6/7

4. [8 *marks*]

(a) Sketch the cardiod given by the polar curve $r = 1 + \sin\theta$, along with the circle $r = 3\sin\theta$.







2/8

(b) Determine the area that lies inside the circle, but outside the cardiod.

r = 1.5   r = 0.5   3 sin θ = 1 + sin θ

A = 2.25π   A = 0.25π $\frac{2\sin\theta}{2} = \frac{1}{2}$

2A = 0.5π $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

5. [6 *marks*] For each of the following, indicate in the space provided whether the statement is true (T) or false (F). No justification for your answer is necessary.

(a) Every bounded monotonic sequence is convergent. Answer: T ✓

(b) Every monotonic convergent sequence is bounded. Answer: T ✓

(c) Every convergent bounded sequence is monotonic. Answer: F ✓

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(d) If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\{a_n\}$ must be convergent. Answer: F T

(e) If $\{a_n\}$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent. Answer: T F

(f) If $a_n > 0$ for all n and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, then $\lim_{n\to\infty} a_n = 0$. Answer: F T

6. [8 *marks*] A sequence is defined recursively by equations $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$.

(a) Prove that the sequence $\{a_n\}$ will converge, by using mathematical induction to show that $\{a_n\}$ is an increasing sequence bounded above by 2.

1) Base Case: $n = \cancel{2}$ 1

Proving $a_n < a_{n+1} < 2$ for all $n \in \mathbb{N}$.

$a_1 = \sqrt{2}$

$a_{1+1} = \sqrt{2 + a_1} = \sqrt{2 + \sqrt{2}}$ ✓

$a_k < 2$

$1.8 \doteq \sqrt{2+\sqrt{2}} > \sqrt{2}$ so $2 > a_2 > a_1$

$\because a_2 > a_1$

$\therefore$ increasing

$a_k < a_{k+1} < 2$ for some $n = k$, $k \in \mathbb{N}$

2) Induction Hypothesis: Assume ~~n=k~~ is true

$a_k = \sqrt{2 + a_{k-1}}$ the assumption is more than the definition of the a_kth term.

$a_{k+1} < 2$

$a_{k+1} < a_{k+2} < 2$

3) Induction Step: Show ~~n=k+1~~ is true

Can't prove bounded by 2 ...

3/8

$a_{k+1} = \sqrt{2 + a_{k+1-1}}$

$= \sqrt{2 + a_k}$

$= \sqrt{2 + \sqrt{2 + a_{k-1}}}$ (from IH)

$\because \sqrt{2 + \sqrt{2 + a_{k-1}}} > \sqrt{2 + a_{k-1}}$ haven't verified the inequality to hold, nor that these values will always be <2.

$\because a_{k+1} > a_k$

Proof by Mathematical induction

$\therefore$ It is an increasing sequence ▨

(b) Determine the value of $\lim_{n\to\infty} a_n$.

(e) If $\{a_n\}$ is convergent, then $\sum_{n=1} a_n$ is convergent. Answer: T F

(f) If $a_n > 0$ for all n and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, then $\lim_{n\to\infty} a_n = 0$. Answer: F T

6. [8 *marks*] A sequence is defined recursively by equations $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$.

(a) Prove that the sequence $\{a_n\}$ will converge, by using mathematical induction to show that $\{a_n\}$ is an increasing sequence bounded above by 2.

Proving $a_n < a_{n+1} < 2$ for all $n \in \mathbb{N}$.

1) Base Case: $n = 1$

$a_1 = \sqrt{2}$

$a_{n+1} = \sqrt{2 + a_1} = \sqrt{2+\sqrt{2}}$ ✓

$a_k < 2$

$1.8 = \sqrt{2+\sqrt{2}} > \sqrt{2}$ ∴ $2 > a_2 > a_1$

∵ $a_2 > a_1$

∴ increasing

$a_k < a_{k+1} < 2$ for some $n = k$, $k \in \mathbb{N}$

2) Induction Hypothesis: Assume ~~n=k is true~~

$a_k = \sqrt{2 + a_{k-1}}$ the assumption is more than the definition of the a_kth term.

$a_{k+1} < 2$

$a_{k+1} < a_{k+2} < 2$

3) Induction Step: Show ~~n=k+1~~ is true

Can't prove bounded by 2 ...

3/8

$a_{k+1} = \sqrt{2 + a_{k+1-1}}$

$= \sqrt{2 + a_k}$

$= \sqrt{2 + \sqrt{2 + a_{k-1}}}$ (from IH)

∵ $\sqrt{2+\sqrt{2+a_{k-1}}} > \sqrt{2 + a_{k-1}}$ haven't verified the inequality to hold, nor that these values will always be <2.

∵ $a_{k+1} > a_k$

∴ It is an increasing sequence ☒ Proof by Mathematical induction

(b) Determine the value of $\lim_{n\to\infty} a_n$.

Since the sequence is upper bounded by 2,

as $\lim_{n\to\infty} a_n = 2$ not necessarily! need to verify this

Answer: As $a_n < a_{n+1} < 2$ for all n ($n \in \mathbb{N}, n \geq 1$), the sequence must converge (Bounded Monotonic sequence theorem).

∴ $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_{n+1} = L$

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2 + a_n}$

$= \sqrt{2 + \lim_{n\to\infty} a_n}$

$L = \sqrt{2 + L}$

$L^2 = 2 + L$

$L = 2$ or -1

See Notes

Since $a_1 = \sqrt{2}$ and $a_n < a_{n+1}$, $L = 2$ is the only value.

7. [6 *marks*] For each of the following, determine whether the sequence $\{a_n\}$ is convergent or divergent. Justify each of your answers.

(a) $a_n = n\cos(n\pi)$

$\cos n\pi$ will always be either 1, or -1

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} |n\cos(n\pi)|$$

$$= \lim_{n\to\infty} \overset{\to\infty}{n}\ \overset{\to 1}{|\cos(n\pi)|}$$

$$= \infty$$

$\therefore$ $\{a_n\}$ it is divergent ✓

(b) $a_n = \dfrac{2^{n+1}}{3^n}$

$$a_n = \frac{2\cdot 2^n}{3^n}$$

$$\lim_{n\to\infty} 2\cdot \underbrace{\left(\tfrac{2}{3}\right)^{n\to\infty}}_{\to 0}$$

$$= 0$$ ✓

$\therefore$ $\{a_n\}$ it is convergent ✓

4/6

(c) $a_n = \dfrac{\sqrt{5n^2+1}}{4-3n}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{5n^2+1}}{4-3n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{5n^2+1}}{\sqrt{(4-3n)^2}}$$ ✓

$$= \lim_{n\to\infty} \sqrt{\frac{5n^2+1}{9n^2-24n+16}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{5+\frac{1}{n^2}}{9-\frac{24}{n}+\frac{16}{n^2}}} = \sqrt{\frac{5}{9}}$$

$\therefore$ $\{a_n\}$ it is convergent ✓

8. [8 *marks*] Consider the telescoping series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$.

Find a formula for the k^{th} partial sum s_k, and use it to find the sum of the series or to show that the series diverges.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$$

$$1 = \frac{A}{2n-1} + \frac{B}{2n+1}$$

$$a_n = \frac{1}{(2n-1)(2n+1)} =$$

$$4n^2-1 = A(2n+1) + B(2n-1)$$

$$S_1 = \frac{1}{(2(1)-1)(2(1)+1)} = \frac{1}{1\cdot 3} = \frac{1}{3}$$

$n = -\frac{1}{2}$ $\quad +1 = -2B$

$B = \frac{1}{2}$

$$S_2 = \frac{1}{(2\cdot 1-1)(2\cdot 1+1)} + \frac{1}{(2\cdot 2-1)(2\cdot 2+1)} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} = \frac{2}{5}$$ ✓

$n = \frac{1}{2}$ $\quad +1 = 2A$

$S_3 = \frac{1}{\ldots} + \ldots$

4/6

(c) $a_n = \dfrac{\sqrt{5n^2+1}}{4-3n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{\sqrt{5n^2+1}}{4-3n}$

$= \lim_{n\to\infty} \dfrac{\sqrt{5n^2+1}}{\sqrt{(4-3n)^2}}$

$= \lim_{n\to\infty} \sqrt{\dfrac{5n^2+1}{9n^2-24n+16}}$

$= \lim_{n\to\infty} \sqrt{\dfrac{5+\frac{1}{n^2}}{9-\frac{24}{n}+\frac{16}{n^2}}} = \sqrt{\frac{5}{9}}$ $\therefore$ ~~It~~ $\sum a_n$ is convergent

8. [8 *marks*] Consider the telescoping series $\sum_{n=1}^{\infty} \dfrac{1}{(2n-1)(2n+1)}$.

Find a formula for the k^{th} partial sum s_k, and use it to find the sum of the series or to show that the series diverges.

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)(2n+1)}$

$1 = \dfrac{A}{2n-1} + \dfrac{B}{2n+1}$

$a_n = \dfrac{1}{(2n-1)(2n+1)} =$

$4n^2-1 = A(2n+1) + B(2n-1)$

$n=-\frac{1}{2}$ $+1=-2B$
$B=\frac{1}{2}$

$n=\frac{1}{2}$ $+1=2A$
$A=\frac{1}{2}$

8/8

$S_1 = \dfrac{1}{(2(1)-1)(2\cdot1+1)} = \dfrac{1}{1\cdot3} = \dfrac{1}{3}$

$S_2 = \dfrac{1}{(2\cdot1-1)(2\cdot1+1)} + \dfrac{1}{(2\cdot2-1)(2\cdot2+1)} = \dfrac{1}{1\cdot3} + \dfrac{1}{3\cdot5} = \dfrac{2}{5}$

$S_3 = \dfrac{1}{1\cdot3} + \dfrac{1}{3\cdot5} + \dfrac{1}{5\cdot7} = \dfrac{3}{7}$

$S_4 = \dfrac{1}{1\cdot3} + \dfrac{1}{3\cdot5} + \dfrac{1}{5\cdot7} + \dfrac{1}{7\cdot9} = \dfrac{4}{9}$

$\frac{1}{3} < \frac{2}{5} < \frac{3}{7} < \frac{4}{9}$

$S_1 < S_2 < S_3 < S_4$

$S_1 = \frac{1}{3}$

$S_2 = \frac{2}{5}$

$S_3 = \frac{3}{7}$

$S_4 = \frac{4}{9}$

$S_k = \dfrac{k}{2k+1}$

$S = \lim_{k\to\infty} S_k$

$= \lim_{k\to\infty} \dfrac{k}{2k+1}$

$= \lim_{k\to\infty} \dfrac{1}{2+1/k}$

$= \frac{1}{2}$

$\therefore$ ~~It~~ $\sum_{n=1}^{\infty} a_n$ converges to $\frac{1}{2}$.
The sum is $\frac{1}{2}$

9. ~~8~~. [15 *marks*] For each of the following series, determine whether it is absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^3+n-1}$

$\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\cos(n)}{n^3+n-1}$ $\quad$ $a_n > 0$ for all n, so cannot use comparison test for Σa_n → but can for $\Sigma |a_n|$

$\displaystyle\underbrace{\frac{|\cos(n)|}{n^3+n-1}}_{a_n} < \frac{1}{n^3+n-1} < \underbrace{\frac{1}{n^3}}_{b_n}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ $\quad$ p-series, $p = 3 > 1$ ✓

∴ convergent

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$|a_n| < b_n$
where b_n is convergent

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^3-n+1}$ is absolutely convergent by Comparison Test.

↳ haven't verified this.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sqrt{n+1}}{n}$

$\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n+1}}{n}$ where $b_n = \frac{\sqrt{n+1}}{n}$

$b_n = \dfrac{\sqrt{n+1}}{n}$

$b_{n+1} = \dfrac{\sqrt{n+2}}{n+1} =$

$\dfrac{\sqrt{n+1}}{n} > \dfrac{\sqrt{n+2}}{n+1}$ verify this (use f' where $f(n) = b_n$)

$b_n > b_{n+1}$

∴ decreasing

$b_n = f(x) = \dfrac{\sqrt{x+1}}{x}$

$f'(x) = \dfrac{\frac{1}{2}(x+1)^{-\frac{1}{2}} \cdot x - \sqrt{x+1}}{x^2}$

$\dfrac{x \cdot \frac{1}{2}(x+1)^{-\frac{1}{2}} - \sqrt{x+1}}{x^2}$

$+ \quad +$

0

$\displaystyle\lim_{n\to\infty} \frac{\sqrt{n+1}}{n}$

$\displaystyle= \lim_{n\to\infty} \sqrt{\frac{n+1}{n^2}}$

$\displaystyle= \lim_{n\to\infty} \sqrt{\frac{\frac{n}{n^2}+\frac{1}{n^2}}{1}}$

$\displaystyle= \lim_{n\to\infty} \sqrt{\frac{1}{n}+\frac{1}{n^2}}$ ✓

$= 0$

$\displaystyle\lim_{n\to\infty} b_n = 0$

2/10

∴ The series is absolutely convergent by AST

~~9~~ 10. [13 *marks*]

(a) Use the Integral Test to determine whether the series $\sum_{n=1}^{\infty} ne^{-n^2}$ is convergent or divergent.

[**Note:** You must first show that the Integral Test can be applied.]

$\sum_{n=1}^{\infty} ne^{-n^2} = \sum_{n=1}^{\infty} a_n$

$a_n = ne^{-n^2}$ f is continuous $[1,\infty)$ ✓ , $f(x)>0$ for $x\in[1,\infty)$ $D: x\in\mathbb{R}$

let $f(x)=xe^{-x^2}$ is ~~differentiable $(1,\infty)$~~ positive

is decreasing function where $a_{n+1}<a_n$

$(n+1)e^{-(n+1)^2} \overset{?}{<} ne^{-n^2}$ need to verify using $f'(x)$.

$a_n = f(x) = xe^{-x^2}$

$f'(x) = e^{-x^2} - 2x^2e^{-x^2}$

$= e^{-x^2}(1-2x^2)$

$\lim_{t\to\infty} \int_1^t f(x)\,dx$

$= \lim_{t\to\infty} \int_1^t xe^{-x^2}\,dx$ $\quad u=-x^2$, $du=-2x\,dx$

$= \lim_{t\to\infty} \frac{-1}{2}\int_{x=1}^{x=t} e^u\,du$ ✓

6/8

$= \lim_{t\to\infty} \frac{-1}{2}e^u\Big]_{x=1}^{x=t}$

$= \frac{-1}{2}\lim_{t\to\infty} e^{-x^2}\Big]_1^t$

$= \frac{-1}{2}\lim_{t\to\infty}\left[e^{-t^2} - e^{-1^2}\right]$

$= \frac{-1}{2}\lim_{t\to\infty}\left[e^{-t^2} - \frac{1}{e}\right]$

$\to 0$ ✓

$= \frac{-1}{2}\left(-\frac{1}{e}\right)$

$= \frac{+1}{2e}$

$\int_1^\infty f(x)\,dx$ is convergent

$\therefore$ The series is convergent ✓

(b) Determine how large the integer N must be to ensure that $\sum_{n=1}^{N} ne^{-n^2}$ is within 0.000 01 of the true value of $\sum_{n=1}^{\infty} ne^{-n^2}$.

$a_1 = 1\cdot e^{-1} = \frac{1}{e} = 0.367879$

$a_3 = 3\cdot e^{-3^2} = \frac{3}{e^9} = 0.000037$

$a_4 = 4\cdot e^{-4^2} = \frac{4}{e^{16}} = 0.00000045$

2/5

$a_4 = S_{n+1}$?

$n=3$

$N\geq$

Also, should know how to use a relevant ~~the~~ integral to determine N.

$N \geq 4$

11. ~~10~~. [12 *marks*]

(a) For the power series given by $f(x) = \sum_{n=0}^{\infty} \frac{2^n (x-3)^n}{n+3}$, determine its radius of convergence and its interval of convergence.

$f(x) = \sum_{n=0}^{\infty} \frac{2^n (x-3)^n}{n+3}$

By Ratio Test

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}(x-3)^{n+1}}{n+1+3} \cdot \frac{n+3}{2^n(x-3)^n}\right|$

$= \lim_{n\to\infty} \left|\frac{2(n+3)}{n+4} \cdot (x-3)\right|$

$= 2 \lim_{n\to\infty} \left|\frac{n+3}{n+4}\right| \cdot |x-3|$ ✓

$= 2|x-3| \cdot \lim_{n\to\infty} \left|\frac{1+\frac{3}{n}}{1+\frac{4}{n}}\right|$

$= 2|x-3|$

$2|x-3| < ~~k~~\ 1$ for convergence

$|x-3| < \frac{1}{2}$

$R = \frac{1}{2}$ ✓

$-\frac{1}{2} < x-3 < \frac{1}{2}$

$\frac{5}{2} < x < \frac{7}{2}$

4/8

at $x = \frac{7}{2}$

$\sum_{n=0}^{\infty} \frac{2^n (\frac{7}{2}-3)^n}{n+3}$ ✓

$= \sum_{n=0}^{\infty} \frac{1}{n+3}$ $\quad a_n = \frac{1}{n+3}$, $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \left|\frac{\frac{1}{n+3}}{\frac{1}{n}}\right| = \lim_{n\to\infty} \left|\frac{n}{n+3}\right| = \lim_{n\to\infty} \left|\frac{1}{1+\frac{3}{n}}\right| = 1 > 0$

Since b_n is divergent (harmonic series ✓), a_n is also divergent by Limit Comparison Test.

$[\frac{5}{3}, \frac{7}{3})$??

The radius of convergence is $\frac{1}{2}$

The interval of convergence is $[\frac{5}{3}, \frac{7}{3})$ 2 2

at $x = \frac{5}{2}$: $\sum_{n=0}^{\infty} \frac{2^n \cdot (\frac{5}{2}-3)^n}{n+3}$

$= \sum_{n=0}^{\infty} \frac{2^n \cdot (-\frac{1}{2})^n}{n+3}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}$ → use Alt. Series Test

$< \frac{(-1)^n}{n}$ where $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent by alternating harmonic series

The 2 series must both be positive to use Comp. Test

by Comparison Test

(b) Determine a power series for $f'(x)$, and state its radius of convergence and its interval of convergence.

$f(x) = \sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{n+3}$

$= \sum_{n=0}^{\infty} (?)$ need starting value to be 0

$= 2|x-3| \cdot \lim_{n\to\infty} \left| \frac{1+\frac{3}{n}}{1+\frac{4}{n}} \right|$

$= 2|x-3|$

$2|x-3| < 1$ for convergence

$|x-3| < \frac{1}{2}$

$R = \frac{1}{2}$ ✓

$\frac{-1}{2} < x-3 < \frac{1}{2}$

$\frac{5}{2} < x < \frac{7}{2}$

6/8

Since b_n is divergent (harmonic series ✓), a_n is also divergent by Limit Comparison Test.

$[\frac{5}{3}, \frac{7}{3})$??

The radius of convergence is $\frac{1}{2}$

The interval of convergence is $[\frac{5}{3}, \frac{7}{3})$ 2 2

at $x = \frac{5}{2}$: $\sum_{n=0}^{\infty} \frac{2^n \cdot (\frac{5}{2}-3)^n}{n+3}$

$= \sum_{n=0}^{\infty} \frac{2^n \cdot (\frac{-1}{2})^n}{n+3}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+3} < \frac{(-1)^n}{n}$ where $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent by alternating harmonic series

→ use Alt. Series Test

by Comparison Test

The 2 series must both be positive to use Comp. Test

(b) Determine a power series for $f'(x)$, and state its radius of convergence and its interval of convergence.

$f(x) = \sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{n+3}$

$f'(x) = \sum_{n=1}^{\infty} \frac{2^n (n)(x-3)^{n-1}}{n+3}$ ✓ $= \sum_{n=1}^{\infty} b_n$

$= \sum_{n=0}^{\infty} (?)$ need starting value to be 0

$R = 3$

interval $= [3]$ x

1/4

$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}(n+1)(x-3)^n}{n+4} \cdot \frac{n+3}{2^n(n)(x-3)^{n-1}} \right|$

$= \lim_{n\to\infty} \left| \frac{2(n+1)(n+3)}{(n+4)(n)} \cdot \frac{1}{(x-3)} \right|$

$= 2\left|\frac{1}{x-3}\right| \cdot \lim_{n\to\infty} \left| \frac{n^2+4n+3}{n^2+4n} \right|$

$= 2\left|\frac{1}{x-3}\right| \cdot \lim_{n\to\infty} \left| \frac{1+\frac{4}{n}+\frac{3}{n^2}}{1+\frac{4}{n}} \right|$

$= 2\left|\frac{1}{x-3}\right| < R$

not req'd, as R = same value as for f(x) and will only need to test at $x = \frac{5}{2}$ for the interval of convergence

$\left|\frac{1}{x-3}\right| < \frac{1}{2}$

$|x-3| > 2$

$x > 5$

x is continuous every where greater than 2.

$R = 2$

$R = \frac{1}{2}$

check end points

at $x = 5$

$\sum_{n=0}^{\infty} \frac{2^n (n)(5-3)^{n-1}}{n+3}$

$= \sum_{n=0}^{\infty} 2^n \cdot n \cdot$

$\sum_{n=1}^{\infty} \frac{2^n (n)(x-3)^{n-1}}{n+3}$

$= \sum_{n=0}^{\infty} \frac{2^{n+1}(n+1)(x-3)^n}{8n+4}$

~~11~~ 12. [4 *marks*] Find the Taylor series for $f(x) = \frac{1}{x}$ centred at $a = 1$. (Assume that $f(x)$ has a power series expansion.)

$f(x) = \frac{1}{x}$ $\quad f(a) = 1$

$f'(x) = \frac{-1}{x^2}$ $\quad f'(a) = -1$

$f''(x) = \frac{+2}{x^3}$ $\quad f''(a) = 2$

$f'''(x) = \frac{-3(2)}{x^4}$ $\quad f'''(a) = -3(2)$

$\vdots$

$f^{(n)}(x) = \frac{(-1)^n \cdot n}{x^{n-1}}$ $n!$

$f^{(n)}(a) = \frac{(-1)^n \cdot n!}{a^{n-1}}$

$= (-1)^n \cdot n!$

$T_n(x) = \frac{(-1)^n \cdot n}{n!} (x \cancel{+} \cancel{a})^n$ $\;1$

where's the summation?!

2/4

$T_n(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!} (x-1)^n$

$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n$ or $\sum_{n=0}^{\infty} (1-x)^n$

~~12~~ 13. [6 *marks*] Recall the Maclaurin series: $f(x) = \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

Using an appropriate Maclaurin series for $g(x) = \sin(x^2)$, find a power series for $\int \sin(x^2)\, dx$ and use your answer to compute $\int_0^{1/2} \sin(x^2)\, dx$ accurate to within 0.000 01.

$f(x) = \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$g(x) = \sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!}$ ✓

$g(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} = \sin(x^2)$ ~~dx~~

$\int g(x)\,dx = \int \sin x^2\, dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}\, dx$

2/6

2/4

$f^{(n)}(x) = \frac{(-1)^n n}{x^{n-1}}$ $n!$

$f^{(n)}(a) = \frac{(-1)^n \cdot n!}{a^{n-1}}$

$= (-1)^n \cdot n!$

$T_n(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!} (x-1)^n$

$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n$ or $\sum_{n=0}^{\infty} (1-x)^n$

13. [6 *marks*] Recall the Maclaurin series: $f(x) = \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

Using an appropriate Maclaurin series for $g(x) = \sin(x^2)$, find a power series for $\int \sin(x^2)\, dx$ and use your answer to compute $\int_0^{1/2} \sin(x^2)\, dx$ accurate to within 0.000 01.

$f(x) = \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$g(x) = \sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!}$ ✓

$g(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} = \sin(x^2)$

2/6

$\int g(x)\,dx = \int \sin x^2\, dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}\, dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!} + C$ ✓

$\int \sin(x^2)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!} + C$

$\int_0^{1/2} \sin(x^2)\,dx = \left[\frac{x^3}{3\cdot 1!} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \dots\right]_0^{\frac{1}{2}}$

$= \left[\left(\frac{1}{2}\right)^3 \cdot \frac{1}{3} - \left(\frac{1}{2}\right)^7 \cdot \frac{1}{7\cdot 6} + \left(\frac{1}{2}\right)^{11} \cdot \frac{1}{11\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} + \dots\right]$

= only need first 3 terms

$= \frac{1}{8\cdot 3} - \frac{1}{128\cdot 42} + \frac{1}{2048\cdot 1320}$

$\boxed{= 0.04148}$