

Instructions:

Non-programmable, non-graphing calculators are permitted. No other aids allowed.

Check that your test paper has no missing, blank, or illegible pages. Note that test questions appear on both sides of the paper.

Answer in the spaces provided.

Show all your work. Insufficient justification will result in a loss of marks.

1. [6 marks] Write your answer to each of the following questions in the space provided. No justification is necessary.

(a) $\lim_{x \rightarrow \infty} 5^{-1/x} = \underline{1}$ ✓

(b) $\lim_{x \rightarrow -\infty} \left(\frac{e}{\pi}\right)^x = \underline{0}$ ✓

$\frac{2.7}{3.1} < 1$ $0 < a < 1$

(c) The domain of the function $f(x) = \sin^{-1} x$ is $\underline{(-1, 1)}$ [1,1]

↳ range of $\sin x$

(d) True or False? If a and b are any real values, then $\ln(ab) = \ln a + \ln b$. True F

include 1, 0, -negative

(e) If $f(x) = |x + 5|$, then f is not differentiable at $x = \underline{-5}$ ✓

(f) If $f(x) = \sin(x)$, then the 103rd derivative, $f^{(103)}(x) = \underline{-\cos(x)}$ ✓

$f^0 = \sin(x)$
 $f^1(x) = \cos(x)$
 $f^2(x) = -\sin(x)$

2. [5 marks] Evaluate the limit: $\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\sin(2\theta) + \tan(3\theta)}$

$\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\sin(2\theta) + \tan(3\theta)} \rightarrow 0$

$\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\sin(2\theta) + \tan(3\theta)} \rightarrow 0$

$f^3(x) = -\cos(x)$
 $f^4(x) = \sin(x)$
 $f^5(x) = \cos(x)$

100 = sin
 101 = cos
 102 = -sin
 103 = -cos

$\lim_{\theta \rightarrow 0} \frac{\cos \theta}{2 \cos(2\theta) + 3 \sec^2(3\theta)}$ ✓

$\lim_{\theta \rightarrow 0} \frac{\cos \theta}{2 \cos(2\theta) + \frac{3}{\cos^2(3\theta)}}$

$\frac{\cos(0)}{2 \cos(0) + \frac{3}{\cos^2(0)}}$ ✓

$\frac{1}{2 + \frac{3}{1}} = \frac{1}{5}$

$\boxed{\frac{1}{5}}$

OR $\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\frac{\sin 2\theta}{2\theta} + \frac{\sin 3\theta}{3\theta} \frac{1}{\cos(3\theta)}}$

OR
 Trig identities

3/6

4/4

3. [7 marks] Consider the function $f(x) = \begin{cases} \ln(-x) - 3x & , x \leq -1 \\ \frac{|x^2 - x - 2|}{x+1} & , x > -1 \end{cases}$

(a) Determine if f is continuous at $x = -1$.

LHL: $\lim_{x \rightarrow -1^-} \ln(-x) - 3x$

$= \ln(-(-1)) - 3(-1)$

$= \ln(1) - (-3)$

$= 0 + 3$

$= 3$

RHL: $\lim_{x \rightarrow -1^+} \frac{|x^2 - x - 2|}{x+1}$

$= \lim_{x \rightarrow -1^+} \frac{|x-2| \cdot |x+1|}{(x+1)}$

$= \lim_{x \rightarrow -1^+} |x-2|$

$= |(-1) - 2|$

$= |-3|$

$= 3$

$\frac{|x^2 - x - 2|}{(x+1)}$
 $= \frac{+(x+1)(x-2)}{(x+1)}$

$= \frac{-(x+1)(x-2)}{(x+1)}$

$= -x + 2$
 $= 3$

and $|x+1| = x+1$
 when $x \rightarrow -1^+$
 as $x+1 > 0$

LHL = RHL $\lim_{x \rightarrow -1} f(x) = 3$

and $f(-1) = 3$

\therefore The function $f(x)$ is continuous at $x = -1$.

(b) Based on our answer to part (a) alone, can we make a conclusion about the differentiability of f at $x = -1$? Explain your answer.

$\lim_{x \rightarrow -1} f(x) = f(-1) = 3$

Rule:

- 1) $f(a)$ exist
 - 2) $\lim_{x \rightarrow a} f(x)$ must exist
 - 3) $\textcircled{1} = \textcircled{2}$
- \rightarrow ex. $\begin{matrix} \bullet < \textcircled{1} < \textcircled{2} \\ \bullet < \textcircled{2} < \textcircled{1} \end{matrix}$ but $1 \neq 2$

no because $f(-1)$ is not defined in the function

when $x > -1$ where the function is $\frac{|x^2 - x - 2|}{x+1}$, $x = -1$ is not defined, therefore it is not differentiable even though it is continuous.

= No, since we're only given that it is continuous, it does not imply that it is diff.

4. [5 marks] Use the Intermediate Value Theorem to show that the equation $e^x - \tan^{-1}(x) = \frac{\pi}{2}$ has a solution in the interval $(0, 1)$.

Verify that the IVT may be applied.

\rightarrow continuity

$f(x) = e^x - \tan^{-1}(x) - \frac{\pi}{2}$
 define a function!

$e^x = \frac{\pi}{2} + \tan^{-1}(x)$

$f(0) = e^0 - \tan^{-1}(0) - \frac{\pi}{2} = 1 - 0 - \frac{\pi}{2}$

$f(1) = e^1 - \tan^{-1}(1) - \frac{\pi}{2}$

to show $\frac{\pi}{2}$ lies in the function's range

5. [8 marks] Find the equation of ^{each} horizontal and ~~vertical~~ asymptote^s of the function

$$f(x) = \frac{4x-1}{2\sqrt{x^2-1}}$$

$$\text{Domain} = (-\infty, -1) \cup (1, \infty)$$

Justify each of your answers by evaluating an appropriate limit.

$$\forall A = \pm 1$$

Positive Infinity

$$\lim_{x \rightarrow +\infty} \frac{4x-1}{2\sqrt{x^2-1}} \times \frac{\sqrt{x^2}}{\sqrt{x^2}}$$

$$= \lim_{x \rightarrow +\infty} \frac{4x/\sqrt{x^2} - 1/\sqrt{x^2}}{2\sqrt{x^2-1}/\sqrt{x^2}}$$

$$= \lim_{x \rightarrow +\infty} \frac{4 - 1/x}{2\sqrt{x^2-1/x^2}} \quad \text{as } \sqrt{x^2} = x \text{ when } x > 0$$

$$= \lim_{x \rightarrow +\infty} \frac{4 - 1/x}{2\sqrt{1-1/x^2}}$$

$$\Rightarrow \frac{4 - 1/\infty}{2\sqrt{1-1/\infty}}$$

$$= \frac{4-0}{2\sqrt{1-0}}$$

$$= \frac{4}{2} \checkmark$$

$$\boxed{= 2}$$

Negative Infinity

$$\lim_{x \rightarrow -\infty} \frac{4x-1}{2\sqrt{x^2-1}} \times \frac{-\sqrt{x^2}}{-\sqrt{x^2}}$$

$$= \lim_{x \rightarrow -\infty} \frac{4x/\sqrt{x^2} - 1/\sqrt{x^2}}{2\sqrt{x^2-1}/\sqrt{x^2}}$$

$$= \lim_{x \rightarrow -\infty} \frac{-4 + 1/x}{-2\sqrt{x^2-1/x^2}}$$

$$= \lim_{x \rightarrow -\infty} \frac{-4 + 1/x}{-2\sqrt{1-1/x^2}}$$

$$= \frac{-4 + 1/\infty}{-2\sqrt{1-1/\infty}}$$

$$= \frac{-4-0}{-2\sqrt{1-0}}$$

$$= \frac{-4}{-2}$$

$$\boxed{= 2} \times \boxed{-2}$$

but $\sqrt{x^2} = -x$ when $x < 0$. $\sqrt{x^2} = -x$

↳ correction

$$x \cdot \frac{1}{x} = \frac{1}{\sqrt{x^2}} \quad \text{ONE ONLY}$$

The equation of horizontal asymptote in this function is $y=2$.

6. [4 marks] Suppose g and h are functions such that $g(x) = \ln(1+h(\sqrt{x}))$,
with $h(2) = 1$ and $h'(2) = 1$. Find $g'(4)$. extra!

$$h(2) = 1$$

$$h'(2) = 1$$

$$g'(4) = ?$$

$$g(x) = \ln(1+h(\sqrt{x}))$$

$$g'(x) = \frac{1}{1+h(\sqrt{x})} \cdot h'(\sqrt{x}) \cdot \frac{1}{2} x^{-1/2}$$

$$g'(4) = \frac{1}{1+h(\sqrt{4})} \cdot h'(\sqrt{4})$$

another chain!

7. [6 marks] Use a linearization, or differentials, to approximate the value of $\sqrt[3]{7.95}$.

$$f(x) \approx L = f(a) + f'(a)(x-a)$$

$a = 8$

$$= \frac{1}{12} + 2(x-8)$$

$$= \frac{1}{12} + 2x - 16$$

$$= 2x - \frac{191}{12}$$

$2x - \frac{191}{12}$

$$f(7.95) \approx L(7.95) = 2(7.95) - \frac{191}{12}$$

$$= -\frac{1}{60} \approx -0.0167$$

$$\sqrt[3]{8} - \sqrt[3]{7.95} = 2 - 1.99582 = 0.00418$$

$$L(8) - L(7.95) = 2 - 0.0167$$

$$= 1.9833$$

$$y = \sqrt[3]{x}$$

$$y = x^{\frac{1}{3}} = f(x)$$

$$y' = \frac{1}{3} x^{-\frac{2}{3}}$$

$$= \frac{1}{3\sqrt[3]{x^2}} \checkmark$$

$$f(8) = \sqrt[3]{8} = 2$$

$$f'(8) = \frac{1}{12}$$

Answer:

~~0.0~~

Let $f(x) = \sqrt[3]{x}$

$$\sqrt[3]{7.95} = f(7.95) \approx L(7.95)$$

$$= f(a) + f'(a)(x-a)$$

8. [5 marks] Prove the derivative formula for the inverse cosine function:

$$\frac{d}{dx} [\cos^{-1} x] = -\frac{1}{\sqrt{1-x^2}}$$

$$y = \cos^{-1} x$$

$$x = \cos y$$

imp diff.

$$1 = -\sin y \cdot \frac{dy}{dx}$$

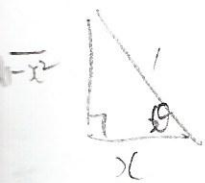
$$\frac{dy}{dx} = \frac{-1}{\sin y}$$

$$= \frac{-1}{\sin(\cos^{-1} x)}$$

$$= \frac{-1}{\sqrt{1-x^2}}$$

= RHS

done on page 7.



$$\cos \theta = x$$

$$\sin \theta = \sqrt{1-x^2}$$

9. [5 marks] Consider $f(x) = (x+1)^{x^2}$, $x > -1$. Determine the equation of the tangent line to $y = f(x)$ at the point $(0, 1)$.

$$y = f(x)$$

$$y = (x+1)^{x^2}$$

$$\ln y = \ln((x+1)^{x^2})$$

$$\ln(y) = x^2 \ln(x+1) \quad \checkmark$$

imp.
diff.

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2x \cdot \ln(x+1) + x^2 \cdot \frac{1}{x+1} \quad \checkmark$$

4/5

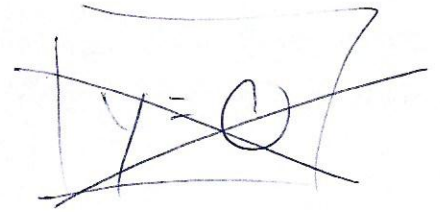
$$\frac{dy}{dx} = y \left(2x \cdot \ln(x+1) + x^2 \cdot \frac{1}{x+1} \right)$$

$$= (x+1)^{x^2} \cdot \left(2x \cdot \ln(x+1) + \frac{x^2}{x+1} \right)$$

$$f'(0) = (0+1)^{0^2} \cdot \left(2(0) \cdot \ln(0+1) + \frac{0^2}{0+1} \right)$$

$$= 1 \cdot (0 \cdot 0 + 0)$$

$$\frac{dy}{dx} = (x+1)^{x^2} \cdot \left(2x \ln(x+1) + \frac{x^2}{x+1} \right)$$



$$f'(0) = 0 \quad \boxed{y=1}$$

So equation of the tangent?

is eqn of tan. line is horiz.

10. [6 marks] Use the Mean Value Theorem (or a resulting theorem/corollary of the MVT) to prove:

$$2 \sin^{-1}(x) - \cos^{-1}(1-2x^2) = 0, \text{ for } x \geq 0.$$

$$\lim_{x \rightarrow a} f(x) = L$$

$$|x-a| < \delta$$

$$|f(x) - L| < \epsilon$$

lim

sin x

11. [10 marks] Consider the function $f(x) = x \cos x - \sin x$, $x \in [0, 2\pi]$. EVT = Extreme Value Theorem

(a) Find the absolute extrema of f on $[0, 2\pi]$.

$$f(x) = x \cos x - \sin x$$

$$f'(x) = \cos x - x \sin x - \cos x = -x \sin x$$

$$f(x) = 0$$

$$0 = -x \sin x$$

$$x = 0, \pi, 2\pi$$

$$f(x) = x \cos x - \sin x$$

$$f(0) = 0$$

$$f(\pi) = -\pi$$

$$f(2\pi) = 2\pi$$

$$f(2\pi) > f(0) > f(\pi)$$

As f is cont on $[0, 2\pi]$ EVT concludes the abs. min value is $-\pi$ (at $x = \pi$) and abs. max value is 2π (at $x = 2\pi$)

\therefore Absolute extrema: Absolute max @ $(2\pi, 2\pi)$
 Absolute min @ $(\pi, -\pi)$
 \rightarrow continuity / EVT

(b) State, with justification, the range of f .

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$$\text{Range} = \{y \in \mathbb{R} \mid -\pi \leq y \leq 2\pi\}$$

as f is cont on $[0, 2\pi]$, so IVT guarantee $f(x) = N$ for any $N \in (-\pi, 2\pi)$

(c) State any relative extrema of f .

$$f'(\pi) = 0$$

$$f(\pi) = -\pi$$

\hookrightarrow relative extrema

local max/min

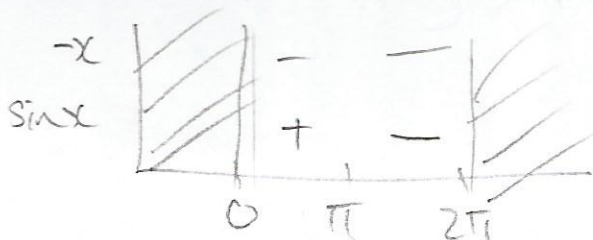
$$f(x) = -x \sin x$$

$$f'(x) = -\sin x + (-x) \cos x$$

$$f'(\pi) = -\sin \pi - \pi \cos \pi = \pi < 0$$

relative min at $x = \pi$

(d) Determine on which interval(s) f is increasing, and on which f is decreasing.



$$f'(x) = -x \sin x$$

extra!

+ve because it's from absolute value

$\epsilon = \delta$ δ

$\delta = \frac{5\epsilon}{4}$

5. [5 marks] Use the formal definition of a limit to prove $\lim_{x \rightarrow 10} (3 - \frac{4}{5}x) = -5$.

$f(x) = y = 3 - \frac{4}{5}x$

$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$
 $= \lim_{h \rightarrow 0} \frac{(3 - \frac{4}{5}(x+h)) - (3 - \frac{4}{5}x)}{h}$

$|x-10| < \delta$
 $|3 - \frac{4}{5}x + 5| < \epsilon$
 $|\frac{4}{5}x + 8| < \epsilon$
 $|\frac{4x+10}{5}| < \epsilon$
 $|x-10| < \frac{5\epsilon}{4}$

$\Rightarrow \lim_{h \rightarrow 0} \frac{-\frac{4h}{5}}{h}$
 $= \lim_{h \rightarrow 0} \frac{-4h}{5h}$
 $= \frac{-4}{5}$

$\lim_{h \rightarrow 0} \frac{1}{h} \left[\left(3 - \frac{4}{5}(x+h) \right) - \left(3 - \frac{4}{5}x \right) \right]$
 $= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{3 - 4(x+h)}{5} - \frac{3 - 4x}{5} \right]$
 $= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-4x - 4h + 4x}{5} \right]$

For any $\epsilon > 0$, there exists a $\delta < 0$ such that $|x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$
 $x \rightarrow 10 \quad \left(3 - \frac{4}{5}(10) \right) = -5$

x	y
10	-5
9.999	-4.9992
9.99	-4.992
9.9	-4.92

As x approaches 10, y or f(x) approaches -5.

$x^2 + 3x + 4$

$2x + 3$

6. [5 marks] Using the definition of the derivative as a limit, carefully show that if $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$. No marks will be given if the limit definition of the derivative is not used.

$2(x+h) + 3 \rightarrow 0$

$2x + 3$

$f(x) = \cos(x)$
 $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$\frac{\cos(x+h) - \cos(x)}{h}$
 $= -\sin(x+h) - \sin(x)$
 $= -\sin(x+h)$

\therefore choose $\delta = \frac{5}{4}\epsilon$, then $|x-10| < \delta$ then $|(3 - \frac{4}{5}x) - (-5)| = \frac{4}{5}|x-10| < \frac{4}{5}\delta = \frac{4}{5}(\frac{5}{4}\epsilon) = \epsilon$
 $\therefore \lim_{x \rightarrow 10} (3 - \frac{4}{5}x) = -5$

$= \lim_{h \rightarrow 0} \frac{1}{h} (f(x+h) - f(x))$

$= \lim_{h \rightarrow 0} \frac{1}{h} (\cos(x+h) - \cos(x))$

$= \lim_{h \rightarrow 0} \frac{1}{h} (\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x))$

$= \lim_{h \rightarrow 0} \frac{1}{h} (\cos(x)\cos(h) - \cos(x) - \sin(x)\sin(h))$

$= \lim_{h \rightarrow 0} \frac{1}{h} (\cos(x)(\cos(h) - 1) - \sin(x)\sin(h))$



$= -\sin(x)$

USE L'Hôpital's Rule and we know $\lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} = 0 = \frac{1 - \cos(h)}{h}$
 $\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1 = \frac{h}{\sin(h)}$

$3/5$

7. [5 marks] Use properties of inverse functions and implicit differentiation to prove the derivative formula for the inverse cosine function: $\frac{d}{dx} [\cos^{-1} x] = -\frac{1}{\sqrt{1-x^2}}$.

$y = \cos^{-1} x$
 $\cos y = x$ for $y \in [0, \pi]$

$\cos \theta = \frac{x}{r}$

9. [6 marks] Use the Mean Value Theorem to prove:

$$\frac{\tan x}{x} > 1, \text{ for } 0 < x < \frac{\pi}{2}.$$

$f(x) = y = \frac{\tan x}{x}$ Let $f(x) = \tan x$

$$f(0) = \frac{\tan(0)}{0} \rightarrow \frac{0}{0}$$

can only be applied to limits

(L'Hôpital)

$$= \frac{\sec^2(x)}{1}$$

$$= \sec^2(x)$$

$$= \frac{1}{\cos^2(0)}$$

$$= \frac{1}{1}$$

$$= 1$$

$$f\left(\frac{\pi}{2}\right) = \frac{\sec^2(x)}{1}$$

$$= \frac{1}{\cos^2\left(\frac{\pi}{2}\right)}$$

$$= \infty$$

$$1 < \frac{\tan x}{x} < \infty$$

$$\left(\frac{\tan x}{x}, \infty\right)$$

or

$$\frac{\tan x}{x} > 1$$

Answer = ① If: f is cont on $(a, b]$
 f is diff on (a, b)

② Then: there must be at least one $c \in (a, b)$ such that
 $f'(c) = \frac{f(b) - f(a)}{b - a}$

Let $f(x) = \tan x$ on the interval $[0, x]$
 where $0 < x < \frac{\pi}{2}$

① f will be continuous on $[0, x]$ as $x \neq \frac{\pi}{2}$

$f(x) = \sec^2(x) \rightarrow$ domain $\text{not } [0, x]$ as $x \neq \frac{\pi}{2}$

② \therefore there exists $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

$$\sec^2(c) = \frac{\tan x - \tan 0}{x}$$

$$\geq 1 = \frac{\tan x}{x}$$

$$\therefore \frac{\tan x}{x} > 1$$

12. [5 marks] When air expands adiabatically (without gaining or losing heat), its pressure P and volume V are related by the equation:

$$PV^{1.4} = C, \text{ where } C \text{ is a constant.}$$

Suppose that the pressure is decreasing at a rate of 10 kPa/min. Determine the rate of change in volume (in cm^3/min) at the point when pressure is 80 kPa, given that $C = 235\,000$.

$$\frac{dP}{dt} = -10$$

$$\frac{dV}{dt} = ?$$

imp diff \int

$$PV^{1.4} = C$$

$$\frac{d}{dt}(V^{1.4}) + P(1.4V^{0.4})\left(\frac{dV}{dt}\right) = 0$$

derivative of a constant = 0

when $P = 80$

$$C = 235\,000$$

$$V^{1.4} = \frac{C}{P}$$

$$V = \sqrt[1.4]{\frac{C}{P}}$$

$$= \sqrt[1.4]{\frac{235\,000}{80}}$$

$$10(300^{1.4}) + 80(1.4(300)^{0.4})\left(\frac{dV}{dt}\right) = 0$$

$$\frac{dV}{dt} = \frac{235\,000 - 10 \cdot 300^{1.4}}{80(1.4(300)^{0.4})}$$

$$= 187.5 \text{ cm}^3/\text{min}$$

3/5