Instructions:

Non-programmable, non-graphing calculators are permitted. No other aids allowed.

Check that your test paper has no missing, blank, or illegible pages. Note that test questions appear on both sides of the paper.

Answer in the spaces provided.

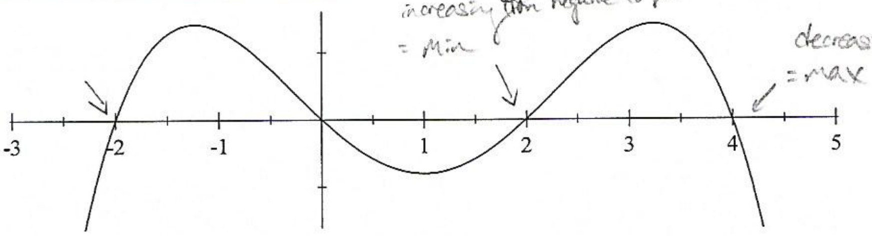
Show all your work. Insufficient justification will result in a loss of marks.

1. [4 *marks*] Write your answer to each of the following questions in the space provided. No justification is necessary.

(a) The graph of the derivative, $f'(x)$, of a function $f(x)$ is given below.

increasing from negative to positive = Min

decreasing from positive to negative = max



Determine all the values of x on the interval $[-3, 5]$ for which f has a relative minimum.

Answer: 1 ✗ $x = -2, 2$

2/4

(b) Given $\int_1^4 f(x)\ dx = 8$, $\int_2^4 f(x)\ dx = 5$ and $\int_1^3 f(x)\ dx = 2$, then $\int_2^3 f(x)\ dx =$ -1 ✓.

$\int_1^4 = 8$ $\int_2^4 = 5$ $\int_1^3 = 2$ $\int_2^4 = 5$ $\int_4^1 = -8$, $\int_1^3 = 2$

(c) When evaluating the limit expression $\lim_{x \to a} f(x)^{g(x)}$, which of the following are indeterminate forms? (Circle your answer(s).) ∞

[✓ 1^∞] ✓ 1^0 ~~[✓ 0^∞]~~ [✓ 0^0] [✓ ∞^0] ✓ ∞^∞

(d) The value of $\int_0^{\sqrt{\pi}} \left(\frac{d}{dx} e^{x^2}\right)\ dx$ is equal to $e^{\pi} - 1$ ✓.

$e^{x^2} \cdot 2x$

2. [4 *marks*] For each of the following, indicate whether the given statement is true (T) or false (F) by writing T or F in the space provided.

(a) Since the function $f(x) = \frac{1}{x}$ is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$, it is impossible to find x_1 and x_2 in the domain of f such that $x_1 < x_2$ and $f(x_1) < f(x_2)$.

Answer: F ✓

2/4

(b) If $f''(x_0) = 0$, then f has a point of inflection at x_0.

concavity must change

Answer: T F

(c) The graph of a function can have, at most, two horizontal asymptotes.

Answer: T ✓

(d) L'Hospital's Rule states that $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, provided the derivatives exist.

Answer: T F

3. [7 *marks*] Evaluate each of the following limits.

(a) $\lim_{x\to 0+} \left[(1+2x)^{1/x}\right]$

$$\lim_{x\to 0^+} \underbrace{(1+2x)}_{\to 0}^{1/x \,\to\, +\infty}$$

$$= 1^{\infty}$$

this is an I.F. (as you indicated in #1c)!)

$$= 1$$

$$\lim_{x\to 0^+} (1+2x)^{\frac{1}{x}}$$

$$= \lim_{x\to 0^+} e^{\ln(1+2x)^{\frac{1}{x}}}$$

$$= \lim_{x\to 0^+} e^{\frac{\ln(1+2x)\,\to 0}{x\,\to 0}}$$

L'Hop

$$= \lim_{x\to 0^+} e^{\frac{\frac{1}{1+2x}(2)}{1}}$$

$$= \lim_{x\to 0^+} e^{\frac{2}{1+2x}}$$

$$= e^2$$

(b) $\lim_{x\to 0+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

3/3

$$= \lim_{x\to 0^+} \left(\underbrace{\frac{1}{x}}_{\infty} - \underbrace{\frac{1}{\sin x}}_{\infty}\right)$$

$$= \infty - \infty \quad ✓$$

$$= \lim_{x\to 0^+} \frac{\sin x - x \,\to 0-0}{x\sin x \,\to 0}$$

L'Hop

$$= \lim_{x\to 0^+} \frac{\cos x - 1 \,\to 0}{\sin x + x\cos x \,\to 0}$$

2nd L'Hôp

$$= \lim_{x\to 0^+} \frac{-\sin x}{\cos x + \cos x - x\sin x} \quad ✓$$

$$= \frac{-\sin(0) \to 0}{1+1-0 \to 2}$$

$$= 0 \quad ✓$$

4. [8 *marks*] Find an equation of the tangent to the curve given below, at the point $\left(0, \frac{1}{2}\right)$.

$$6\sin^{-1}(x+y) = \pi(x^2 + e^{xy})$$

$$6\arcsin(x+y) = \pi\left(x^2 + e^{xy}\right)$$

imp. diff.

$$6\left(\frac{1}{\sqrt{1-(x+y)^2}}\left(1+\frac{dy}{dx}\right)\right) = \pi\left(2x + e^{xy}\left(x\frac{dy}{dx} + y\right)\right) \quad ✓$$

$$\frac{6}{\sqrt{1-(x+y)^2}} + \frac{6}{\sqrt{1-(x+y)^2}}\frac{dy}{dx} = 2\pi x + \pi x e^{xy}\frac{dy}{dx} + \pi y e^{xy}$$

$$\frac{\frac{\pi}{2} - \frac{6}{\sqrt{\frac{3}{4}}}}{}$$

5. [10 *marks*] Consider the function $y = f(x) = \dfrac{10 \ln x}{x^2}$. A+

(a) Determine if the graph of f has any vertical asymptotes and any horizontal asymptotes, justifying your answers by calculating relevant limits.

Vert. Asym.

$\lim_{x \to 0^+} \dfrac{10 \ln x}{x^2} \begin{matrix} \to -\infty \\ \to 0^+ \end{matrix}$

L'Hôp ↳ $= \lim_{x \to 0^+} \dfrac{10 \frac{1}{x}}{2x} = -\infty$

$= \lim_{x \to 0^+} \dfrac{10}{2x^2}$

$= \infty$

$\lim_{x \to 0^-} \dfrac{10 \ln x}{x^2} \begin{matrix} \to \text{DNE} \\ \to 0 \end{matrix}$

There is a vertical asymptote $x=0$. ✓

(only on the right side)

$\lim_{x \to +\infty} \dfrac{10 \ln x}{x^2} \begin{matrix} \to \infty \\ \to \infty \end{matrix}$

L'Hôp ↳ $= \lim_{x \to +\infty} \dfrac{10}{2x^2}$ -

$= 0$ ✓

Domain for $\ln x = \{x \in \mathbb{R} \mid x > 0\}$

There is a horizontal asymptote $y=0$.

Since $\lim x \to 0^-$, there are no x-values less than 0. No H.A. on left side.

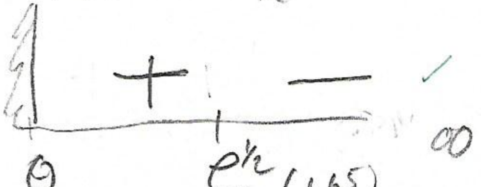
(b) Determine on which intervals the graph of f is increasing, and on which intervals it is decreasing.

You may use the fact that $f'(x) = \dfrac{10(1 - 2\ln x)}{x^3}$.

$f'(x) = \dfrac{10(1-2\ln x)}{x^3}$

$f'(x) = 0$
when $x = e^{1/2}$

$1 - 2\ln x = 0$
$\frac{1}{2} = \ln x$ ✓
$x = e^{1/2}$
$\ln(1) = 0$



$f'(x) = \text{DNE}$
when $x = 0$

9/10

The graph f is increasing on $(0, e^{1/2})$ ✓
decreasing on $(e^{1/2}, \infty)$

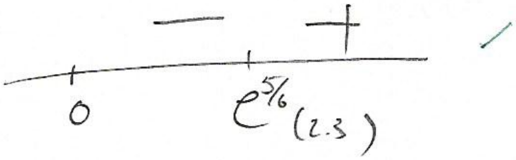
(c) Determine on which intervals the graph of f is concave up, and on which it is concave down.

You may use the fact that $f''(x) = \dfrac{10(6\ln x - 5)}{x^4}$.

$f''(x) = \dfrac{10(6\ln x - 5)}{x^4}$

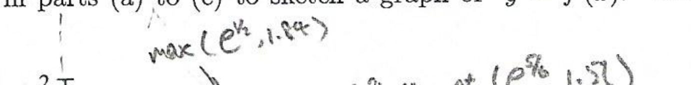
$6\ln x - 5 = 0$
$\ln x = \frac{5}{6}$
$x = e^{5/6}$

$f''(0) = \text{DNE}$
$f''(x) = 0$
when $x = e^{5/6}$ ✓



The graph f is concave up when $x \in (e^{5/6}, \infty)$ ✓
concave down when $x \in (0, e^{5/6})$

(d) Use the information found in parts (a) to (c) to sketch a graph of $y = f(x)$. Label any key features of the graph.



$\lim_{x\to0^+} \frac{10}{2x^2}$

$= \infty$

$\lim_{x\to0^-} \frac{10\ln x}{x^2} \begin{matrix}\to DNE \\ \to 0\end{matrix}$

$\lim_{x\to+\infty} \frac{10\ln x}{x^2} \begin{matrix}\to\infty \\ \to\infty\end{matrix}$

L'Hôp $\lim_{x\to+\infty} \frac{10}{2x^2}$

$= 0$ ✓

Domain for $\ln x = \{x \in \mathbb{R} \mid x > 0\}$

There is a horizontal asymptote $y=0$.

Since $\lim x \to 0^-$, there are no x-values less than 0. No H.A. on left side.

(b) Determine on which intervals the graph of f is increasing, and on which intervals it is decreasing.

You may use the fact that $f'(x) = \frac{10(1-2\ln x)}{x^3}$.

$f'(x) = \frac{10(1-2\ln x)}{x^3}$

$f'(x)=0$ when $x=e^{1/2}$

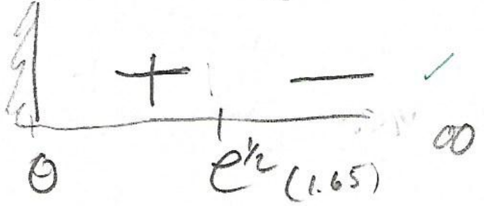
$f'(x)=DNE$ when $x=0$

$1-2\ln x = 0$

$\frac{1}{2} = \ln x$ ✓

$x = e^{1/2}$

$\ln(1)=0$



9/10

The graph f is increasing on $(0, e^{1/2})$ ✓
decreasing on $(e^{1/2}, \infty)$

(c) Determine on which intervals the graph of f is concave up, and on which it is concave down.

You may use the fact that $f''(x) = \frac{10(6\ln x - 5)}{x^4}$.

$f''(x) = \frac{10(6\ln x - 5)}{x^4}$

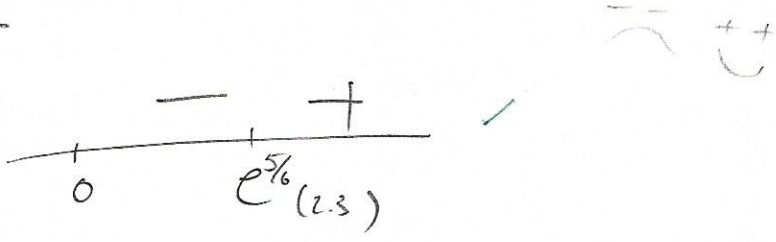
$6\ln x - 5 = 0$

$\ln x = \frac{5}{6}$

$x = e^{5/6}$

$f''(0) = DNE$

$f''(x) = 0$ when $x = e^{5/6}$ ✓



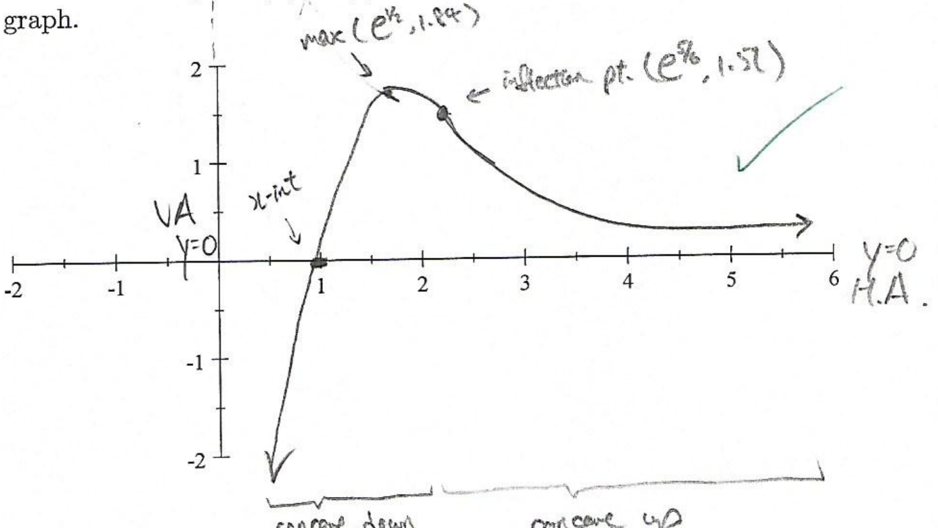
The graph f is concave up when $x \in (e^{5/6}, \infty)$ ✓
concave down when $x \in (0, e^{5/6})$

(d) Use the information found in parts (a) to (c) to sketch a graph of $y = f(x)$. Label any key features of the graph.

$f(e^{1/2}) \doteq 1.84$

$f(e^{5/6}) \doteq 1.57$

$x\text{-int} = (1, 0)$



6. [10 *marks*] Use the Intermediate Value Theorem to show that the equation

$$\sin(x) - 2x = 1 - \frac{1}{\sqrt{2}}$$

has at least one solution in the open interval $\left(-\frac{\pi}{4}, 0\right)$. Then use Rolle's Theorem to prove that, in fact, the equation has exactly one solution.

define $f(x) = \dots$

$\sin(0) - 2(0) = 0$

$\sin\left(\frac{-\pi}{4}\right) - 2\left(\frac{-\pi}{4}\right) =$

$-\frac{\sqrt{2}}{2} + \frac{\pi}{2} = \frac{\pi - \sqrt{2}}{2}$

3/10

$\doteq 0.86$

$1 - \frac{1}{\sqrt{2}} \doteq 0.29$

→ need continuity of $f(x)$ on $[-\frac{\pi}{4}, 0]$

~~is continuous~~ ✓ OK!

Since $f(x) = 1 - \frac{1}{\sqrt{2}}$ exists in between $f(\frac{-\pi}{4})$ and $f(0)$ (which are $\frac{\pi - \sqrt{2}}{2}$ and 0) respectively). It shows that there is at least one solution ✓ in the interval $(\frac{-\pi}{4}, 0)$.

IVT: If ① f is cont' on $[a, b]$

② $f(a) < L < f(b)$ or $f(b) < L < f(a)$

Then there exists at least one $c \in (a, b)$ such that $f(c) = L$

Rolle: If ① f is cont' on $[a, b]$

② f is diff' on (a, b)

© $f(a) = f(b)$

Then there is at least one $c \in (a, b)$ such that $f'(c) = 0$

$f(x) = \sin(x) - 2x - 1 + \frac{1}{\sqrt{2}}$

$\sin(\frac{-\pi}{4}) - 2(\frac{-\pi}{4}) =$

$-\frac{\sqrt{2}}{2} + \frac{\pi}{2} = \frac{\pi - \sqrt{2}}{2}$

3/10

$\doteq 0.86$

$1 - \frac{1}{\sqrt{2}} \doteq 0.29$

→ need continuity of f(x) on $[-\frac{\pi}{4}, 0]$

~~is continuous~~ ✓ OK!

Since $f(x) = 1 - \frac{1}{\sqrt{2}}$ exists in between $f(\frac{-\pi}{4})$ and $f(0)$ (which are $\frac{\pi-\sqrt{2}}{2}$ and 0) respectively). It shows that there is at least one solution in the interval $(\frac{-\pi}{4}, 0)$. ✓

<u>IVT</u>: If ① f is cont' on [a, b]

② $f(a) < L < f(b)$ or $f(b) < L < f(a)$

Then there exists at least one $c \in (a,b)$ such that $f(c) =$

<u>Rolle</u>: If ① f is cont' on [a,b]

② f is diff' on (a,b)

③ $f(a) = f(b)$

Then there is at least one $c \in (a,b)$ such that $f'(c) = 0$

$f(x) = \sin(x) - 2x - 1 + \frac{1}{\sqrt{2}}$

<u>IVT</u> i) ① f has domain $(-\infty, \infty)$ ∴ f is cont on $[\frac{-\pi}{4}, 0]$

② $f(-\frac{\pi}{4}) > 0$, $f(0) < 0$ ∴ by IVT, $f(x) = 0$ for at least one value $c \in (\frac{-\pi}{4},$

<u>Rolle</u> ii) assume there are two roots at $x = a$ & $x = b$ $(a \neq b)$

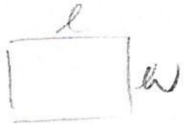
① f will be cont on [a,b] (as Domain = $\mathbb{R}$)

② $f'(x) = \cos x - 2$ → domain = $\mathbb{R}$ ∴ f is diff on (a,b)

③ $f(a) = f(b) = 0$

∴ It can't have more than one sol'n.

7. [10 *marks*] A rectangular box with an open top is to have a volume of 48 m^3. The length of the base is to be twice the width. The material for the base costs \$8 per square metre. Material for the sides will cost \$6 per square metre. Find the cost of producing the cheapest such box.



$V = l(w)(h)$ $l = 2w$

$48 = (2w^2)h$

$24 = w^2 h$

$h = \frac{24}{w^2}$

1/10

base =

lw

cost of base: $8(lw)$

$C = 8(2w)(w) + 6(6w(\frac{24}{w^2}))$

$= 16w^2 + \frac{864}{w}$

Cost of side:

④2 × $(6(w(h)))$

$+ 2 \times (6(l(h)))$

put together to get the function for total cost

$6(2lh + 2wh)$

$6(4wh + 2wh)$

$86(wh)$

$C' = 32w + 864(-1)w^{-2}$

$C' = 32w - \frac{864}{w^2}$

$C' = 0$

$0 = 32w - \frac{864}{w^2}$

$864 = 32w^3$

$27 = w^3$

$w = 3$

$$\begin{array}{r} 64 \\ 27\overline{)1728} \\ \underline{162} \\ 108 \end{array}$$

$$\begin{array}{r} 36 \\ \underline{24} \\ 720 \\ \underline{144} \\ 864 \end{array}$$

$$\begin{array}{r} 27 \\ 32\overline{)864} \\ \underline{64} \\ 224 \\ \underline{214} \end{array}$$

$C'' = 32 - 1728w^{-3}$

$C''(3) = 32 - \frac{1728}{27} < 0$

$= -32 < 0$

min

$C(3) = 16(3)^2 + \frac{864}{3}$

$= 144 + 288$

$= \$432$

$$\begin{array}{r} 32 \\ \underline{9} \\ 288 \end{array}$$

$$\begin{array}{r} 16 \\ \underline{9} \\ 144 \end{array}$$

$48 = lwh$

$48 = 2w^2h$

$8lw + 6(2)wh + 6(2)lh$

$8lw + 12wh + 12(2w)h$

8. [8 *marks*] Use a Riemann sum with a regular partition to determine $\int_2^6 (x+3)^2 \, dx$.

$\lim_{n \to \infty} \frac{4}{n} \sum_{i=1}^{n} \overset{f(2+\frac{4}{n}i)}{(x+3)^2}$

$\frac{n(n-1)}{2}$?

$= \sum_{i=1}^{n}$

$A = \int_a^b f(x)dx$

$w \qquad h$

$= \lim_{n \to \infty} \left[\sum_{i=1}^{n} \left(\frac{b-a}{n}\right) \cdot \left(f\left(a + \frac{b-a}{n} i\right)\right) \right]$

$x_0 = a + \frac{b-a}{n} i$

$= 2 + \frac{4}{n} i$

$= \lim_{n \to \infty} \left[\sum_{i=1}^{n} \left(\frac{6-2}{n}\right) \left(f\left(2 + \frac{4}{n} i\right)\right) \right]$

$= \lim_{n \to \infty} \left[\sum_{i=1}^{n} \left(\frac{4}{n}\right) \left(2 + \frac{4}{n} i + 3\right)^2 \right]$

$= \lim_{n \to \infty} \frac{\cancel{4}}{n} \left[\sum_{i=1}^{n} \left(25 + \frac{40}{n} i + \frac{16}{n^2} i^2\right) \right]$

$= \lim_{n \to \infty} \frac{4}{n} \left[25(n) + \frac{40}{n} \left(\frac{n(n+1)}{2}\right) + \frac{16}{n^2} \left(\frac{n(2n+1)(n+1)}{6}\right) \right]$

$= \lim_{n \to \infty} 100 + \frac{80(n+1)}{n} + \frac{32(2n+1)(n+1)}{3n^2}$

$= \lim_{n \to \infty} 100 + 80 \frac{n+1}{n} + \frac{32}{3} \frac{2n^2+3n+1}{n^2}$

$= \lim_{n \to \infty} 100 + 80 + \frac{32}{3}(2)$

$= \cancel{2\#2} \; \frac{604}{3}$

9. [7 *marks*] Let $\mathcal{R}$ denote the region bounded between the curves $y = \frac{x}{2}$ and $y = \sqrt{x}$. Find the volume of the solid obtained by rotating $\mathcal{R}$ about the line $x = 4$.

Area: $f(x) - g(x)$

$f(x) = \sqrt{x} \rightarrow x = y^2 \Rightarrow 4 - \sqrt{x}$

$4 - y^2$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left(\frac{4}{n}\right)\left(2+\frac{4}{n}i+3\right)^2\right]$$

$$= \lim_{n\to\infty}\frac{4}{n}\left[\sum_{i=1}^{n}\left(25+\frac{40}{n}i+\frac{16}{n^2}i^2\right)\right]$$

$$= \lim_{n\to\infty}\frac{4}{n}\left[25(n)+\frac{40}{n}\left(\frac{n(n+1)}{2}\right)+\frac{16}{n^2}\left(\frac{n(2n+1)(n+1)}{6}\right)\right]$$

$$= \lim_{n\to\infty} 100+\frac{80(n+1)}{n}+\frac{32(2n+1)(n+1)}{3n^2}$$

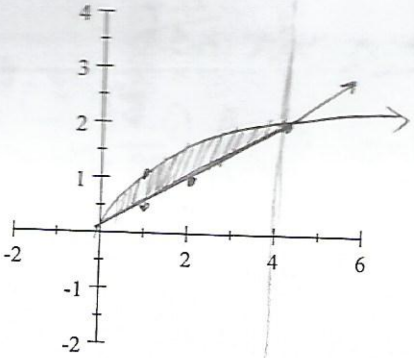
$$= \lim_{n\to\infty} 100+80\,\frac{n+1}{n}+\frac{32}{3}\,\frac{2n^2+3n+1}{n^2}$$

$$= \lim_{n\to\infty} 100+80+\frac{32}{3}(2)$$

$$= \cancel{212}\ \frac{604}{3}$$

9. [*7 marks*] Let $\mathcal{R}$ denote the region bounded between the curves $y=\dfrac{x}{2}$ and $y=\sqrt{x}$. Find the volume of the solid obtained by rotating $\mathcal{R}$ about the line $x=4$.

4/7



Area: $f(x)-g(x)$

$f(x)=\sqrt{x} \rightarrow x=y^2 \Rightarrow 4-\sqrt{x}$ $\qquad 4-y^2$

$g(x)=\frac{x}{2} \rightarrow x=2y \Rightarrow 4-\frac{x}{2}$ $\qquad 4-2y$

Horizontal slices → so need to integrate with respect to y.

these are y-values so expression should be in same variable! →

$$\xcancel{\int_0^2 \pi\left[(4-\sqrt{x})^2-\left(4-\frac{x}{2}\right)^2\right]dx}$$

$$\xcancel{=\int_0^2 \pi\left(16-8\sqrt{x}+x-16+4x-\frac{x^2}{4}\right)dx}$$

$$\xcancel{=\int_0^2 \pi\left(-8\sqrt{x}+5x-\frac{x^2}{4}\right)dx}$$

$$\xcancel{\pi\left(-8\left(\frac{2}{3}\right)x^{\frac{3}{2}}+\frac{5}{2}x^2-\frac{1}{12}x^3\right)\Big]_0^2}$$

$\left(4-\frac{x}{2}\right)^2$

$16-8\left(\frac{x}{2}\right)$

$\int_0^{2}$

$A=\int_0^4\left(\sqrt{x}-\frac{x}{2}\right)dx$

or

$=\int_0^4(2y-y^2)dy$ ✓

$$=\pi\left(\frac{-16}{3}\sqrt{8}+10-\frac{2}{3}\right)$$

$$\boxed{=\frac{-16\pi}{3}\sqrt{8}+\frac{28\pi}{3}\text{ units}^3}$$

this would be negative!!

✓method

Answer:

$\int_0^{2}\pi\left((4-y^2)^2-(4-2y)^2\right)$

$=\frac{32\pi}{5}$

10. [5 *marks*] Use a substitution to evaluate $\int_1^6 x\sqrt{x+3}\,dx$.

let $u = x+3$

$du = dx$

$x = u-3$

$$\int_1^6 x\sqrt{x+3}\,dx$$

$$= \int_{x=1}^{x=6} (u-3)\sqrt{u}\,du$$

$$= \int_1^6 u\sqrt{u} - 3\sqrt{u}\,du$$

$$= \int_1^6 u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\,du$$

$$= \frac{2}{5}u^{\frac{5}{2}} - 3\left(\frac{2}{3}\right)u^{\frac{3}{2}} \Big]_{x=1}^{x=6}$$

these are x-values, cannot sub in for u

$$= \left[\frac{2}{5}6^{\frac{5}{2}} - 2(6)^{\frac{3}{2}}\right] - \left[\frac{2}{5} - 2\right]$$

$$= \frac{72\sqrt{6}}{5} - 12\sqrt{6} - \frac{8}{5}$$

$$\boxed{\doteq 4.29}$$

4/5

$u = x$ $dv = \sqrt{x+3}$

$du = dx$ $v = (x+3)^{\frac{3}{2}}\left(\frac{2}{3}\right)$

$\frac{2}{3}x(x+3)^{\frac{3}{2}} - \int \frac{2}{3}(x+3)^{\frac{3}{2}}\,dx$

$u = 9$

$u = 4$

$$\left(\frac{2}{5}(9)^{\frac{5}{2}} - 2(9)^{\frac{3}{2}}\right) - \left(\frac{2}{5}(4)^{\frac{5}{2}} - 2(4)^{\frac{3}{2}}\right)$$

11. [7 *marks*] Use integration by parts to evaluate $\int_1^2 \tan^{-1}\sqrt{x-1}\,dx$.

$$\int_1^2 \tan^{-1}(\sqrt{x-1})\,dx$$

$u = \tan^{-1}\sqrt{x-1}$ $dv = dx$

$du = \frac{1}{x^2}\,dx$ $v = x$

? ✗

$$= \frac{1}{1+(\sqrt{x-1})^2} \cdot \frac{1}{2}(x-1)^{-1/2}\,dx$$

$$= \frac{1}{2x\sqrt{x-1}}\,dx$$

$$uv - \int v\,du$$

$$= \arctan(\sqrt{x-1})(x)\Big|_1^2 - \int_1^2 x\frac{1}{x^2}\,dx$$

$$= \arctan(\sqrt{x-1})(x)\Big|_1^2 - \int_1^2 \frac{1}{x}\,dx$$

$$= x\cdot\arctan(\sqrt{x-1}) - \ln|x|\Big]_1^2$$

$\frac{1}{1+x-1}$

5/7

$$= 2(\arctan(1) - \ln(2)) - (\arctan(0) - 0)$$

$$= \left(\frac{\pi}{2} - \ln 2\right) - (0)$$

✓ method

$$= \frac{\pi}{2} - \ln 2$$

$$\arctan(\sqrt{x-1})(x)\Big]_1^2 - \int_1^2 \frac{x}{2x\sqrt{x-1}}\,dx$$

12. [*8 marks*] Determine $\int \frac{9x^2+3x+13}{(x-2)(x^2+2x+3)}\,dx$ by the method of partial fraction expansion.

Degree $P < Q$ for $\frac{P(x)}{Q(x)}$

$$\frac{9x^2+3x+13}{(x-2)(x^2+2x+3)} = \frac{A}{(x-2)} + \frac{Bx+C}{(x^2+2x+3)}$$ ✓

$$9x^2+3x+13 = A(x^2+2x+3) + (Bx+C)(x-2)$$

$x=2$

$$9(4)+6+13 = A(4+4+3) + 0$$

$$55 = 11A$$

$$A = 5$$ ✓

10/12

$x=0$

$$13 = 3A + [B(0)+C](-2)$$

$$= 3A + (-2C)$$

$$13 = 15 - 2C$$

$$-2 = -2C$$ ✓

$$C = 1$$

$x=1$

$$9+3+13 = A(1+2+3) + (B+C)(-1)$$

$$25 = 6A - B - C$$

$$25 = 30 - B - 1$$

$$-4 = -B$$ ✓

$$B = 4$$

$$\int \frac{9x^2+3x+13}{(x-2)(x^2+2x+3)}\,dx$$

$$= \int \frac{5}{x-2} + \frac{4x+1}{x^2+2x+3}\,dx$$

$$= \int \frac{5}{x-2}\,dx + \int \frac{4x+1}{x^2+2x+3}\,dx$$ ✓ → complete the square first, do a substitution, and then separate into 2 integrals.

$$= 5\ln|x-2| + \int \frac{4x}{x^2+2x+3}\,dx + \int \frac{1}{}$$

$$\int \frac{4x}{x^2+2x+3}\,dx$$

$$= \int 4x \left(\frac{1}{x^2+2x+3}\right) dx$$

$$= 4\int \frac{x}{x^2+2x+3}\,dx$$

$$= 4\int \frac{x}{(x+1)^2+2}$$

13. [6 *marks*] Show that the two-variable function $u = \sin(x - at) + \ln(x + at)$ satisfies the wave equation $u_{tt} = a^2 u_{xx}$ (where a is a constant).

$a \frac{1}{x+at}$

$U = \sin(x-at) + \ln(x+at)$

L.S. $Ut = \cos(x-at)(-a) + \frac{1}{x+at}(a)$ ✓

$= -a\cos(x-at) + \frac{a}{x+at}$

$Utt = -a^2\sin(x-at) - a^2(x+at)^{-2}$ ✓

$= a^2(-\sin(x-at) - (x+at)^{-2})$

6/6

$Ux = \cos(x-at) + \frac{1}{x+at}$ ✓

R.S. $Uxx = -\sin(x-at) - (x+at)^{-2}$ ✓

$$L.S. = \underbrace{-a^2(\sin(x-at)+(x+at)^{-2})}_{Utt} = R.S.\ a^2\underbrace{(-\sin(x-at)-(x+at)^{-2})}_{Uxx}$$ ✓

L.S=R.S.

14. [8 *marks*] Find all critical points and determine the relative extrema (if any) of

$$f(x, y) = x^3 - 3xy + 3y^2 - 9y.$$

Use the Second Derivative Test to justify your conclusions.

$f(x,y) = x^3 - 3xy + 3y^2 - 9y$

$fx(x,y) = 3x^2 - 3y$ ✓

$fy(x,y) = -3x + 6y - 9$

~~$fxy(x,y) = -3$~~

$fxx(x,y) = 6x$

$fyy = 6$

$fxy = -3$

only use this if $D(x,y) > 0$ where $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$

$fxx(-1,1) = -6 \xleftarrow{X}$ local max

$3x^2 - 3y = 0$

$3x^2 = 3y$

$-3x + 6y - 9 = 0$

$6y - 9 = 3x$

$6y = 3x + 9$

$3y = \frac{3x+9}{2}$

$3x^2 = \frac{3x+9}{2}$

$6x^2 = 3x + 9$

$3(\frac{9}{4}) = 3y$

$y = \frac{9}{4}$

5/8

$U_{tt} = -a^2\sin(x-at) - a^2(x+at)^{-2}$ ✓

$= a^2(-\sin(x-at) - (x+at)^{-2})$

6/6

$U_x = \cos(x-at) + \frac{1}{x+at}$ ✓

R.S. $U_{xx} = -\sin(x-at) - (x+at)^{-2}$ ✓

L.S. $: \underbrace{-a^2(\sin(x-at)+(x+at)^{-2})}_{U_{tt}} = $ R.S. $a^2 \underbrace{(-\sin(x-at)-(x+at)^{-2})}_{U_{xx}}$ ✓

L.S=R.S.

14. [8 *marks*] Find all critical points and determine the relative extrema (if any) of

$$f(x,y) = x^3 - 3xy + 3y^2 - 9y.$$

Use the Second Derivative Test to justify your conclusions.

$f(x,y) = x^3 - 3xy + 3y^2 - 9y$

$f_x(x,y) = 3x^2 - 3y$ ✓

$f_y(x,y) = -3x + 6y - 9$

~~$f_{xy}(x,y) = -3$~~ $f_{yy} = 6$

$f_{xx}(x,y) = 6x$ $f_{xy} = -3$

only use this if $D(x,y) > 0$ where $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$

$f_{xx}(-1,1) = -6 \leftarrow$ local ~~min~~ ✗

$f_{xx}(\frac{3}{2},\frac{9}{4}) = 9 \leftarrow$ local max

~~local min : $(-1,1)$~~

~~local max : $(\frac{3}{2},\frac{9}{4})$~~

$(-1,1)$ = saddle pt.

$3x^2 - 3y = 0$

$3x^2 = 3y$

$-3x + 6y - 9 = 0$

$6y - 9 = 3x$

$6y = 3x + 9$

$3y = \frac{3x+9}{2}$

$3x^2 = \frac{3x+9}{2}$

$6x^2 = 3x + 9$

$2x^2 = x + 3$

$2x^2 - x - 3 = 0$ ✓

$(x+1)(2x-3) = 0$

$x = -1 \rightarrow y = 1$

$x = \frac{3}{2} \rightarrow y = \frac{9}{4}$ ✓

$3(\frac{9}{4}) = 3y$

$y = \frac{9}{4}$

5/8

$D = f_{xx} f_{yy} - (f_{xy})^2$

$-6 \quad 6 - 9 < 0$

$9 \quad 6 - 9 > 0$