[*10 marks*] **1.** Without using L'Hôpital's Rule, evaluate the following limits if they exist:

(a) $\lim\limits_{x\to 3} \dfrac{x^2-2x-3}{x-3}$

$= \lim\limits_{x\to 3} \dfrac{(x-3)(x+1)}{(x-3)}$

$= \lim\limits_{x\to 3} (x+1)$

$= 3+1$

$= 4$

$\therefore$ This limit exists at 4 when x approaches 3.

(b) $\lim\limits_{x\to 4^-} \dfrac{2x-8}{|x-4|}$

$= \lim\limits_{x\to 4^-} \dfrac{2(x-4) \to -}{|x-4| \to +}$

$= \lim\limits_{x\to 4^-} \dfrac{-2(x-4)}{|x-4|}$

$= -2$

$\therefore$ This limit exists at -2 when x approaches 4 from the left

(c) $\lim\limits_{x\to 0} \dfrac{\sqrt{x+16}-4}{x}$

$= \lim\limits_{x\to 0} \dfrac{\sqrt{x+16}-4}{x} \times \dfrac{\sqrt{x+16}+4}{\sqrt{x+16}+4}$

$= \lim\limits_{x\to 0} \dfrac{x+16-16}{x(\sqrt{x+16}+4)}$

$= \lim\limits_{x\to 0} \dfrac{x}{x(\sqrt{x+16}+4)}$

$= \lim\limits_{x\to 0} \dfrac{1}{\sqrt{x+16}+4}$

$= \dfrac{1}{\sqrt{16}+4} = \dfrac{1}{8}$

$\therefore$ This limit exists at $\frac{1}{8}$ when x approaches 0.

(d) $\lim\limits_{x\to\infty} \dfrac{\sqrt{4x^4+3}}{x^2-5x^{3/2}+2}$

$\sqrt{x^4} = x^2$

$= \lim\limits_{x\to\infty} \dfrac{\sqrt{4x^4+3}}{x^2-5x^{3/2}+2} \times \dfrac{\frac{1}{\sqrt{x^4}}}{\frac{1}{\sqrt{x^4}}}$

[5 marks] **2.** $f(x)$ is defined as $f(x) = \left\{ \begin{array}{ccc} e^{Ax} & \text{when} & x \geq 0 \\ 3x + B & \text{when} & x < 0 \end{array} \right\}$

3

For what values of A and B will $f(x)$ be continuous and differentiable everywhere.

f(x) is continuous $[-\infty, \infty]$
f(x) is differentiable $(-\infty, \infty)$

$e^{Ax} \geq 0$ $3x + B < 0$

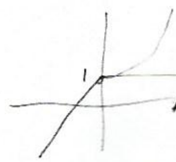
$e^{Ax} = 3x + B$

when $x = 0$

$\hookrightarrow e^{0 \cdot A} = 3 \cdot 0 + B$ $e^{Ax} = 3x + 1$

$e^{0} = 0 + B$

$B = 1$



A must be ≥ 0
$A \neq 0$ since $e^0 = 1$
$f(x) = 1$ can't be differentiable.

$A \geq 0$
$B = 1$

∴ The value of A must be any positive integer where $(A \geq 0)$ the value of B must be 1 in order for the function to be continuous and differentiable everywhere.

If negative A = not differentiable = cusp

[3 marks] **3.** (a) State the limit definition of the derivative.

3

let f(x) be the function

$y = f(x)$

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

$|x - a| < \delta$
$|f(x) - L| < \varepsilon$

[5 marks] (b) Using the limit definition of the derivative, compute the derivative for the function: $f(x) = x^2 + 3x - 2$

5

$f(x) = x^2 + 3x - 2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) - 2 - (x^2 + 3x - 2)}{h}$$

$$= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) + 3x + 3h - 2 - x^2 - 3x + 2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 + 3h}{h}$$

[*12 marks*] **4.** For each of following, find $y' = \dfrac{dy}{dx}$ using any method you wish. Leave your answer in terms of x or x and y. Do **NOT** spend time simplifying your answers.

(a) $y = x^3 2^x$

$\frac{dy}{dx} = y' = 3x^2 \cdot 2^x + x^3 \cdot 2^x \cdot \ln 2$ ✓

3

$\therefore \frac{dy}{dx} = 3x^2 + 2^x + x^3 \cdot 2^x \cdot \ln 2$

(b) $y = \dfrac{2\tan(x)}{1 + x^2}$

$\frac{dy}{dx} = y' = \dfrac{2\sec^2(x)(1+x^2) - 2\tan(x)(2x)}{(1+x^2)^2}$ ✓

3

(c) $y = (\cos(x))^{1/x}$

$y = (\cos(x))^{\frac{1}{x}}$

~~2~~ 3

$\ln y = \ln\left((\cos(x))^{\frac{1}{x}}\right)$

imp. diff.

$\ln y = \frac{1}{x} \cdot \ln(\cos(x))$

$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{-1}{x^2} \cdot \ln(\cos(x)) + \frac{1}{x} \cdot \frac{1}{\cos(x)} \cdot (-\sin(x))$ ✓

$\frac{dy}{dx} = (\ldots)\left[\frac{-\ln(\cos(x))}{\ldots} \ldots -\sin(x) \ldots\right]$

(b) $y = \frac{2\tan(x)}{1+x^2}$

$$\frac{dy}{dx} = y' = \frac{2\sec^2(x)(1+x^2) - 2\tan(x)(2x)}{(1+x^2)^2}$$

3

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imp. diff.

$$\ln y = \frac{1}{x} \cdot \ln(\cos(x))$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{-1}{x^2} \cdot \ln(\cos(x)) + \frac{1}{x} \cdot \frac{1}{\cos(x)} \cdot (-\sin(x))$$

$$\frac{dy}{dx} = y\left[\frac{-\ln(\cos(x))}{x^2} + \frac{-\sin(x)}{x\cos(x)}\right]$$

$(\cos(x))^{\frac{1}{x}}$

(d) $y^2 = e^{x+y} + \sin(x^2)$

$$y^2 = e^{x+y} + \sin x^2$$

imp. diff.

$$2y \cdot \frac{dy}{dx} = e^{x+y} \cdot \left(1 + \frac{dy}{dx}\right) + \cos(x^2) \cdot (2x)$$

3

$$2y \cdot \frac{dy}{dx} = \left[e^{x+y} + (e^{x+y})\left(\frac{dy}{dx}\right)\right] + 2x(\cos(x^2))$$

$$2y \cdot \frac{dy}{dx} - e^{x+y} \cdot \frac{dy}{dx} = e^{x+y} + 2x(\cos(x^2))$$

$$(2y - e^{x+y})\left(\frac{dy}{dx}\right) = e^{x+y} + 2x(\cos(x^2))$$

$$\frac{dy}{dx} = \frac{e^{x+y} + 2x(\cos(x^2))}{(2y - e^{x+y})}$$

11/12

Over

[*5 marks*] **5.** Using L'Hôpital's Rule, if it is needed, evaluate the following limits.

(a) $\lim_{x \to 0} \frac{x}{\log_3(x+1)}$

$= \lim_{x\to 0} \frac{x}{\log_3(x+1)} \begin{matrix} \to 0 \\ \to 0 \end{matrix}$

$\log_3(x+1) = \frac{\ln(x+1)}{\ln 3}$

$= \lim_{x\to 0} \frac{x}{\left(\frac{\ln(x+1)}{\ln 3}\right)}$

$= \lim_{x\to 0} \frac{x \ln 3}{\ln(x+1)} \begin{matrix} \to 0 \\ \to 0 \end{matrix}$

$\frac{-1}{x^2}$

L'Hôp $= \lim_{x\to 0} \frac{\ln 3}{\frac{1}{x+1}\cdot(1)}$

$= \lim_{x\to 0} \frac{\ln 3}{1}$

$\boxed{= \ln 3}$ $\therefore \lim_{x\to 0} \frac{x}{\log_3(x+1)} = \ln 3$

(b) $\lim_{x \to \infty} x \sin(1/x^2)$

$= \lim_{x\to\infty} x \sin\left(\frac{1}{x^2}\right)$ $\sin(0) = 0$

$x \to \infty$, $\frac{1}{x^2} \to 0$

Indeterminant form: $0 \cdot \infty$

$= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x^2}\right)}{1/x}$

L'Hôp $= \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x^2}\right)\left(\frac{-2}{x^3}\right)}{\left(\frac{-1}{x^2}\right)}$

$= \lim_{x\to\infty} \frac{-2\cos\left(\frac{1}{x^2}\right)}{-x^3} \cdot x^2$

$= \lim_{x\to\infty} \frac{2\cos\left(\frac{1}{x^2}\right)}{x} \begin{matrix} \to 2 \\ \to \infty \end{matrix}$

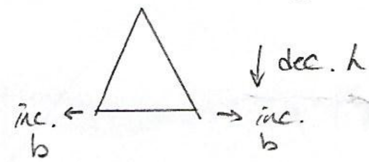
$\boxed{= 0}$

$\therefore \lim_{x\to\infty} x \sin\left(\frac{1}{x^2}\right) = 0$

[*7 marks*] **6.** A pile of sand forms a cone ($V = \frac{1}{3}\pi r^2 h$). Over time, rainfall wears away at the cone, causing the height of the cone to decrease, while maintaining a conical shape. None of the sand is blowing away, so the volume remains constant. The current height of the cone is $2m$ and its base radius if $3m$. If the base radius of the cone is increasing at a rate of $0.075m/hr$, then how fast is the height decreasing?

7

Given: $V = \frac{1}{3}\pi r^2 h$



$h = 2m$ $V = \frac{1}{3}(\pi)(3^2)(2)$

$r = 3m$ $V = 6\pi$

current volume

$\frac{dr}{dt} = 0.075 m/hr$

$h = \frac{V}{\frac{1}{3}\pi r^2}$

Find: $\frac{dh}{dt} = ?$

$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = \frac{\pi}{3}(2r)\left(\frac{dr}{dt}\right)(h) + \frac{\pi}{3} r^2 \left(\frac{dh}{dt}\right)$

$0 = \frac{\pi}{3}(2\cdot 3)(0.075)(2) + \frac{\pi}{3}(3)^2\left(\frac{dh}{dt}\right)$

$= \frac{3\pi}{10} + 3\pi \frac{dh}{dt}$

$\frac{-3\pi}{10} = 3\pi \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{-1}{}$

$\frac{6\pi}{dt} = 0$

$6\pi = \frac{\pi}{3} r^2 h$

$18 = r^2 h$

$0 = 2r \cdot \frac{dr}{dt}(h) + r^2 \cdot \frac{dh}{dt}$

$= 6 \cdot 0.075(2) + 9 \cdot \frac{dh}{dt}$

[*13 marks*] **7.** Consider the following function on $[-3, 3]$, along with its derivatives:

$$f(x) = \frac{-x}{x^2+4} \qquad f'(x) = \frac{x^2-4}{(x^2+4)^2} \qquad f''(x) = \frac{2x(12-x^2)}{(x^2+4)^3}$$

(a) Find the roots/zeroes and vertical asymptotes of $f(x)$ (if there are any).

(b) Find the critical points of $f(x)$ and then use a sign chart to find the intervals where f is increasing/decreasing (if there are any).

(c) Find any local maximum and local minimum values (if there are any) and justify by the first or second derivative test. Determine the absolute maximum and minimum values on $[-3, 3]$.

(d) Find the intervals of concavity (if there are any).

(e) Use the information in (a)-(d) to sketch $f(x)$ on $[-3, 3]$ labeling any important points. Draw your sketch neatly on the back page if you need more space.

$f(x) = \frac{-x}{x^2+4}$ $\quad f'(x) = \frac{x^2-4}{(x^2+4)^2}$ $\quad f''(x) = \frac{2x(12-x^2)}{(x^2+4)^3}$

(1,0).
x,y
+.

a) roots : $f(x) = 0$

$f(x) = \frac{-x}{x^2+4} = 0$

$x = 0$

roots = $(0,0)$

Vertical Asymptote :

2

as $\lim_{x\to a} f(x) = \infty$

$\lim_{x\to 0} f(x) = 0$

$\lim_{x\to 2} f(x) = \frac{-1}{4}$

No Vertical Asymptotes

b) Critical points:

using $f'(x) = \frac{x^2-4}{(x^2+4)^2}$

① if $x = 0$

② if x = undefined $\rightarrow x \in \mathbb{R}$, x is defined everywhere

① $f(x) = 0$ when $f'(x) = \frac{x^2-4}{(x^2+4)^2}$

$f(2) = 0$
$f(-2) = 0$

$x = \pm 2$

Critical points @ $(-2, \frac{1}{4})$ and $(2, \frac{-1}{4})$

$+ \quad - \quad +$

c) critical points : $x = \pm 2$

2nd derivative test $\begin{cases} f''(2) = 0.0625 > 0 & \text{min} \\ f''(-2) = -0.0625 < 0 & \text{max} \end{cases}$

Local maximum @ $(-2, \frac{1}{4})$

3

Local minimum @ $(2, \frac{-1}{4})$

$f(x)$ is cont on $[-3, 3]$
is diff' on $(-3, 3)$

$f(-3) = 0.23$

$f(3) = -0.23$

$f(-2) = \frac{1}{4} = 0.25$ ← max

$f(2) = \frac{-1}{4} = -0.25$ ← min

As $f(x)$ is cont' on $[-3, 3]$, Extreme Value Theorem concludes that the absolute maximum value is at $\frac{1}{4}$ (at $x = -2$) and the absolute minimum value is at $\frac{-1}{4}$ (at $x = 2$).

Absolute max = $(-2, \frac{1}{4})$
Absolute min = $(2, \frac{-1}{4})$ in $D = [-3, 3]$

d) $f''(x) = 0$ or undefined

$x = 0, \pm\sqrt{12}$

(a) Find the roots/zeroes and vertical asymptotes of $f(x)$ (if there are any).

(b) Find the critical points of $f(x)$ and then use a sign chart to find the intervals where f is increasing/decreasing (if there are any).

(c) Find any local maximum and local minimum values (if there are any) and justify by the first or second derivative test. Determine the absolute maximum and minimum values on $[-3, 3]$.

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(e) Use the information in (a)-(d) to sketch $f(x)$ on $[-3, 3]$ labeling any important points. Draw your sketch neatly on the back page if you need more space.

$f(x) = \frac{-x}{x^2+4}$ $f'(x) = \frac{x^2-4}{(x^2+4)^2}$ $f''(x) = \frac{2x(12-x^2)}{(x^2+4)^3}$

(1,0)
x, y
+.

a) roots : $f(x) = 0$

$f(x) = \frac{-x}{x^2+4} = 0$

$x = 0$

roots = (0,0)

2

Vertical Asymptote :

as $\lim_{x\to a} f(x) = \infty$

$\lim_{x\to 0} f(x) = 0$

$\lim_{x\to 2} f(x) = \frac{-1}{4}$

No Vertical Asymptotes

b) Critical points:

using $f'(x) = \frac{x^2-4}{(x^2+4)^2}$

① if $x = 0$

② if x = undefined → $x \in \mathbb{R}$, x is defined everywhere

① $f(x) = 0$ when $f'(x) = \frac{x^2-4}{(x^2+4)^2}$

$f(2) = 0$
$f(-2) = 0$

$x = \pm 2$

Critical points @ $(-2, \frac{1}{4})$ and $(2, \frac{-1}{4})$

+		−		+
	−2		2	

$f'(-3) = \frac{5}{169}$ + ← max

$f'(0) = \frac{-4}{16}$ −

$f'(3) = \frac{5}{169}$ + ← min

} 1st derivative test for c)

3

$f(x)$ is increasing in intervals of $(-\infty, -2) \cup (2, \infty)$

$f(x)$ is decreasing in intervals of $(-2, 2)$

c) Critical points: $x = \pm 2$

2nd derivative test $\begin{cases} f''(2) = 0.0625 > 0 & \text{min} \\ f''(-2) = -0.0625 < 0 & \text{max} \end{cases}$

Local maximum @ $(-2, \frac{1}{4})$

3 Local minimum @ $(2, \frac{-1}{4})$

$f(x)$ is cont on $[-3, 3]$

is diff' on $(-3, 3)$

$f(-3) = 0.23$

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$f(-2) = \frac{1}{4} = 0.25$ ← max

$f(2) = \frac{-1}{4} = -0.25$ ← min

As $f(x)$ is cont' on $[-3, 3]$, Extreme Value Theorem concludes that the absolute maximum value is at $\frac{1}{4}$ (at $x = -2$) and the absolute minimum value is at $\frac{-1}{4}$ (at $x = 2$).

Absolute max : $(-2, \frac{1}{4})$

Absolute min = $(2, \frac{-1}{4})$ in $D = [-3, 3]$

d) $f''(x) = 0$ or undefined

$x = 0, \pm\sqrt{12}$

3

+		−		+		−
	$-\sqrt{12}$		0		$\sqrt{12}$	

outside Domain

$f''(-4) = 0.004$

$f''(-1) = -0.176$

$f''(1) = 0.176$

$f''(4) = -0.004$

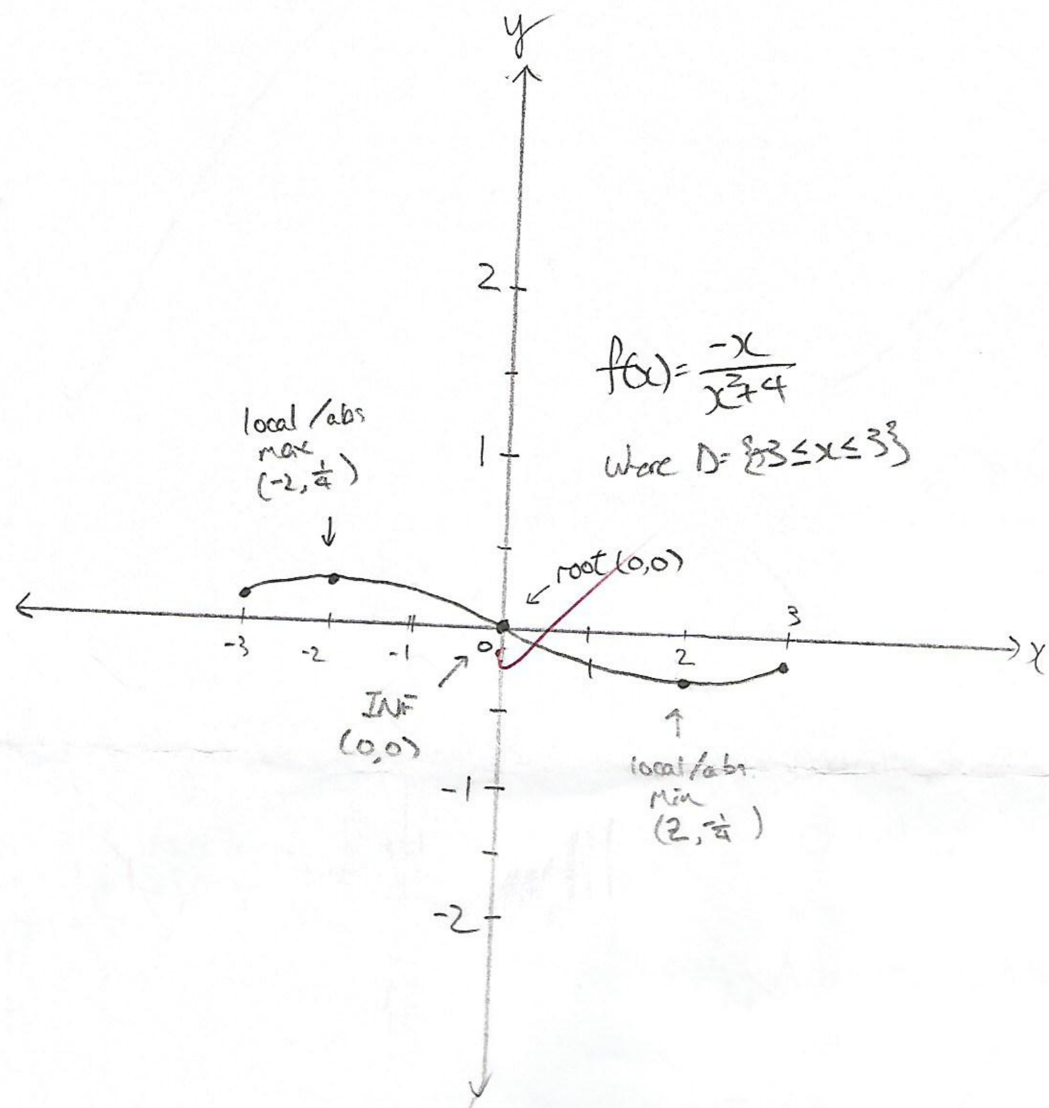
The function f is concave up in interval of $(0, 3)$

/13

Extra space, if needed

e)

2



INF pt of $\pm\sqrt{12}$ is out of Domain