MATH 1800 Summer 2015

Midterm Solutions

1. For each statement below, determine if it is true or false. Provide a brief explanation justifying your answer.

   (a) (1 mark) For any set S, $\emptyset \in S$.

   (b) (1 mark) For any set S, $\emptyset \in \mathcal{P}(S)$.

   Solution:

   (a) This is false. For example, the set $S = \{1, 2, 3\}$ does not contain the empty set as a member.

   (b) This is true. The elements of the power set of S are the subsets of S. The empty set is a subset of any set S, and so the empty set is a member of $\mathcal{P}(S)$.

2. (2 marks) Does $A - B = \emptyset$ imply $B - A = \emptyset$? If so, prove it. If not, give a counterexample.

   Solution: This is false. For example, let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$. Then $A - B = \emptyset$ but $B - A = \{3\}$. Any sets A and B with $A \subseteq B$ and $A \neq B$ will yield a valid counterexample.

3. Let $P(x, y)$ be the open sentence "x loves y", where the domain of x and y is the set of all people. For parts (a) and (b), translate each of the following statements into English. Try to make your translations as natural sounding as possible, but be sure what you write is not ambiguous.

   (a) (1 mark) $\forall x \exists y P(y, x)$

   (b) (2 marks) $\forall x \big((\forall y P(y, x)) \Rightarrow (\exists z P(x, z)) \big)$

   (c) (2 marks) Use logical equivalences seen in class to rewrite

   $$\sim\forall x \big((\forall y P(y, x)) \Rightarrow (\exists z P(x, z)) \big)$$

   so that all negation symbols immediately precede the component statements.

   (d) (2 marks)Rewrite the following English sentence using only P, the universal and existential quantifier, and logical connectives: "If a person does not love oneself, then they love no one".

   Solution:

   (a) A rough translation is "for any person x, there is a person person y such that y loves x." A better translation is "each person is loved by somebody."

   (b) A rough translation is "for any person x, if for all people y, y loves x, then there is a person z such that x loves z." A much better translation is "If a person is loved by everyone, then that person loves somebody."

(c) We have

$$\begin{aligned}\sim\forall x((\forall y P(y,x)) \Rightarrow (\exists z P(x,z))) &\equiv \exists x \sim((\forall y P(y,x)) \Rightarrow (\exists z P(x,z)))\\ &\equiv \exists x((\forall y P(y,x)) \wedge (\sim\exists z P(x,z)))\\ &\equiv \exists x((\forall y P(y,x)) \wedge (\forall z \sim P(x,z))).\end{aligned}$$

(d) This may be written as

$$\forall x\left((\sim P(x,x)) \Rightarrow (\forall y \sim P(x,y))\right).$$

The following answers are logically equivalent, and hence are also valid.

$$\forall x\left(P(x,x) \vee (\forall y \sim P(x,y))\right)$$

$$\sim\exists x\left((\sim P(x,x)) \wedge (\exists y P(x,y))\right)$$

There are many other equivalent forms.

4. (5 marks) Use any valid method to show that $(\sim P) \Rightarrow \big((\sim Q) \Rightarrow P\big)$ and $P \vee Q$ are logically equivalent.

Solution: We may use a truth table to show the two statements are equivalent. We have

P	Q	$\sim P$	$\sim Q$	$(\sim Q) \Rightarrow P$	$(\sim P) \Rightarrow ((\sim Q) \Rightarrow P)$	$P \vee Q$
T	T	F	F	T	T	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	F	F	F

Since the columns for $(\sim P) \Rightarrow ((\sim Q) \Rightarrow P)$ and $P \vee Q$ are the same, they are logically equivalent.

Alternate Solution: We can use previously seen logical equivalences to deduce the required equivalence. We do this below.

$$\begin{aligned}(\sim P) \Rightarrow ((\sim Q) \Rightarrow P) &\equiv (\sim P) \Rightarrow ((\sim\sim Q) \vee P) \quad (\text{since } A \Rightarrow B \equiv (\sim A) \vee B)\\ &\equiv (\sim\sim P) \vee ((\sim\sim Q) \vee P) \quad (\text{since } A \Rightarrow B \equiv (\sim A) \vee B)\\ &\equiv P \vee (Q \vee P) \quad (\text{since } \sim\sim A \equiv A)\\ &\equiv P \vee (P \vee Q) \quad (\text{since } A \vee B \equiv B \vee A)\\ &\equiv (P \vee P) \vee Q \quad (\text{since } A \vee (B \vee C) \equiv (A \vee B) \vee C)\\ &\equiv P \vee Q \quad (\text{since } A \vee A \equiv A).\end{aligned}$$

5. As seen in class, any number is congruent modulo n to its remainder when divided by n. If we let $n = 7$, since there are only 7 possible remainders when dividing by 7 $(0, 1, 2, 3, 4, 5$ and $6)$, every integer is congruent to one of these remainders. For each number a below, find a number $b \in \{0, 1, 2, 3, 4, 5, 6\}$ that is congruent to a modulo 7.

(a) (0.5 marks) $a = 45$

(b) (0.5 marks) $a = -24$

(c) (0.5 marks) $a = 777$

(d) (0.5 marks) $a = 4321$

Solution: Remember that when working modulo 7, we can add or subtract integer multiples of 7 and the result will be congruent to the original number.

(a) Since $42 = 6 \cdot 7$ and $45 - 42 = 3$, we have $45 \equiv 3 \pmod 7$, and so $b = 3$.

(b) Since $28 = 4 \cdot 7$ and $-24 + 28 = 4$, we have $-24 \equiv 4 \pmod 7$, and so $b = 4$.

(c) Since 777 is divisible by 7, $777 \equiv 0 \pmod 7$, and so $b = 0$.

(d) Since $4200 = 100 \cdot 6 \cdot 7$ and $4321 - 4200 = 121$, we have $4321 \equiv 121 \pmod 7$. Since $70 = 10 \cdot 7$ and $121 - 70 = 51$, we have $121 \equiv 51 \pmod 7$. Since $49 = 7 \cdot 7$ and $51 - 49 = 2$, we have $51 \equiv 2 \pmod 7$, and so $b = 2$.

6. (8 marks) Let $x, y \in \mathbb{Z}$. Prove that xy is even if and only if x is even or y is even. For this question, do not use any previously seen results (just use the definitions of even and odd numbers).

Proof. We first show the forward direction with a proof by contraposition. Suppose x is odd and y is odd. Then $x = 2k + 1$ and $y = 2\ell + 1$ for some integers k and ℓ. We have

$$\begin{aligned} xy &= (2k+1)(2\ell+1) \\ &= 4k\ell + 2k + 2\ell + 1 \\ &= 2(2k\ell + k + \ell) + 1. \end{aligned}$$

Since $2k\ell + k + \ell$ is an integer, xy is odd.
We now show the reverse direction. Suppose x is even or y is even. Without loss of generality, assume x is even. Then $x = 2k$ for some integer k. We have

$$xy = 2ky.$$

Since ky is an integer, xy is even. □

Note: If in the reverse direction a student does not use "without loss of generality" and instead shows both cases (case 1: x is even, and case 2: y is even), they will still get full marks (as long as their work is correct).

7. (6 marks) Let A and B be sets. Prove that $\overline{A} - B = \overline{A \cup B}$.

Proof. We first show $\overline{A} - B \subseteq \overline{A \cup B}$. Let $x \in \overline{A} - B$. Then $x \in \overline{A}$ and $x \notin B$. That is, $x \notin A$ and $x \notin B$. By DeMorgan's Law (logic version), we see that it's not the case that $x \in A$ or $x \in B$. That is, it's not the case that $x \in A \cup B$, so $x \in \overline{A \cup B}$. Thus $\overline{A} - B \subseteq \overline{A \cup B}$.
We now show $\overline{A \cup B} \subseteq \overline{A} - B$. Let $x \in \overline{A \cup B}$. Then it's not the case that $x \in A \cup B$. That is, it's not the case that $x \in A$ or $x \in B$. By DeMorgan's Law (logic version), we see that $x \notin A$ and $x \notin B$. That is, $x \in \overline{A}$ and $x \notin B$. Thus $x \in \overline{A} - B$. We've shown $\overline{A \cup B} \subseteq \overline{A} - B$. □

In the following alternate solution, we use DeMorgan's Law for sets rather than DeMorgan's Law for logic.

Alternate Proof. We first show $\overline{A} - B \subseteq \overline{A \cup B}$. Let $x \in \overline{A} - B$. Then $x \in \overline{A}$ and $x \notin B$. That is, $x \in \overline{A}$ and $x \in \overline{B}$. Thus $x \in \overline{A} \cap \overline{B}$. By DeMorgan's Law (set version), $\overline{A} \cap \overline{B} = \overline{A \cup B}$, and so $x \in \overline{A \cup B}$. Thus $\overline{A} - B \subseteq \overline{A \cup B}$.
We now show $\overline{A \cup B} \subseteq \overline{A} - B$. Let $x \in \overline{A \cup B}$. Then by DeMorgan's Law (set version), $x \in \overline{A} \cap \overline{B}$. Thus $x \in \overline{A}$ and $x \notin B$, and so $x \in \overline{A} - B$. We've shown $\overline{A \cup B} \subseteq \overline{A} - B$. □

In the following alternate solution, we use previously seen set identities to establish the result. We will use the fact that $A - B = A \cap \overline{B}$, as well as DeMorgan's Law (for sets).

Alternate Proof. We have

$$\begin{aligned} \overline{A} - B &= \overline{A} \cap \overline{B} \\ &= \overline{A \cup B}. \end{aligned}$$

□

That was quick!

8. (6 marks) Let $n \in \mathbb{Z}$. Prove that $3n^3 \not\equiv 1 \pmod 9$.

Proof. We use a proof by contradiction. Suppose $3n^3 \equiv 1 \pmod 9$. Then $9 \mid (3n^3 - 1)$. Thus $3n^3 - 1 = 9k$ for some integer k. Rearranging, we have

$$1 = 3n^3 - 9k = 3(n^3 - 3k).$$

Since $n^3 - 3k$ is an integer, this implies $3 \mid 1$, which is a contradiction. □

9. (8 marks) Use Mathematical Induction to prove that for all $n \in \mathbb{N}$ with $n \geq 2$,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + ... + \frac{1}{(n-1)(n)} = \frac{n-1}{n}.$$

Proof. We proceed by induction. When $n = 2$, the left hand side is $\frac{1}{1 \cdot 2} = \frac{1}{2}$, and the right hand side is $\frac{2-1}{2} = \frac{1}{2}$, and so the result holds.
We now let $k \in \mathbb{N}$, $k \geq 2$ and assume

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + ... + \frac{1}{(k-1)(k)} = \frac{k-1}{k}.$$

We must show

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + ... + \frac{1}{(k)(k+1)} = \frac{k}{k+1}.$$

We have

$$\begin{aligned}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{(k-1)(k)}+\frac{1}{(k)(k+1)} &= \frac{k-1}{k}+\frac{1}{k(k+1)} \\ &= \frac{1}{k}\left(\frac{(k-1)(k+1)}{k+1}+\frac{1}{k+1}\right) \\ &= \frac{1}{k}\left(\frac{k^2}{k+1}\right) \\ &= \frac{k}{k+1}.\end{aligned}$$

By the principle of mathematical induction, the result holds for all $n \in \mathbb{N}$, $n \geq 2$. $\square$

10. (4 marks) Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$. **Disprove**: If $a \mid bc$, then $a \mid b$ or $a \mid c$.

Solution: There are many counterexamples. For a counterexample to be valid, the premise must be true (that is, $a \mid bc$) but the conclusion must be false (that is, $a \nmid b$ *and* $a \nmid c$, by DeMorgan's Law). For example, $8 \mid (4 \cdot 6)$, but $8 \nmid 4$ and $8 \nmid 6$. (The statement is actually true if we require that a be a prime number).