Question 1:

Consider a continuous-time system which has input of signal $x(t)$ and output of $y(t) = x(t)u(t)$.

a) Is this system time invariant? Justify your answer.
b) Is this system linear? Justify your answer.

Part a:

To prove that the system is time invariant, we should show that for any input $x_1(t)$ and any time shift t_0, we have $y_2(t) = y_1(t-t_0)$, where $x_1(t) \to y_1(t)$, $x_2(t) \to y_2(t)$ and $x_2(t) = x_1(t-t_0)$. Otherwise, the system is time variant.

Proof is as follows:

$$y_1(t) = x_1(t)u(t) \Rightarrow y_1(t-t_0) = x_1(t-t_0)u(t-t_0)$$
$$y_2(t) = x_2(t)u(t) = x_1(t-t_0)u(t)$$

Answer: Therefore, $y_2(t) \neq y_1(t-t_0)$ and the system is time variant.

Part b:

To prove that the system is linear, we should show that for any input $x_1(t)$ and $x_2(t)$ and any scalar a and b, we have $y_3(t) = ay_1(t) + by_2(t)$, where $x_1(t) \to y_1(t)$, $x_2(t) \to y_2(t)$, $x_3(t) \to y_3(t)$ and $x_3(t) = ax_1(t) + bx_2(t)$. Otherwise, the system is non-linear.

Proof is as follows:

$$y_3(t) = x_3(t)u(t) = \{ax_1(t) + bx_2(t)\}u(t) = ax_1(t)u(t) + bx_2(t)u(t) = ay_1(t) + by_2(t)$$

Answer: Therefore, $y_3(t) = ay_1(t) + by_2(t)$ and the system is linear.

Question 2:

Consider a discrete-time system which has input of signal $x[n]$ and output of $y[n] = \cos\left[\frac{\pi}{4}x[n]\right]$.

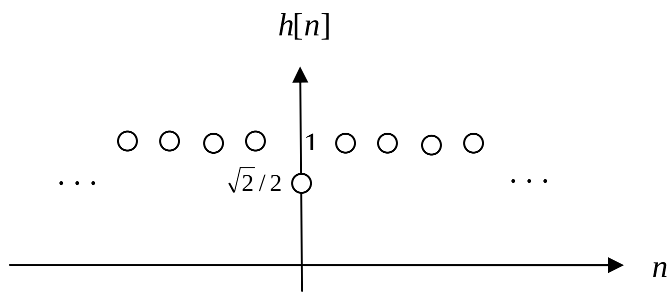
a) Evaluate and draw the impulse response of the above system.
b) If the input to the system is $x[n] = \frac{n^2}{2}$, determine whether the output of the system $y[n]$ is periodic. If $y[n]$ is periodic, find its fundamental period and fundamental frequency.

Part a:

By definition, the impulse response is $h[n] = y[n]\big|_{x[n]=\delta[n]} = \cos\left[\frac{\pi}{4}\delta[n]\right]$ and therefore,

Answer:

$$h[n] = \begin{cases} \cos\left[\frac{\pi}{4}\times 1\right] = \frac{\sqrt{2}}{2} & n = 0 \\ \cos\left[\frac{\pi}{4}\times 0\right] = 1 & n \neq 0 \end{cases}$$



Part b:

To prove that $y[n]$ is periodic, we should find a positive integer number N such that for any n, $y[n] = y[n+N]$.

$y[n] = \cos\left[\frac{\pi}{4}\times\frac{n^2}{2}\right] = \cos\left[\frac{\pi n^2}{8}\right]$ and $y[n+N]\cos\left[\frac{\pi(n+N)^2}{8}\right]$

If k and l are integer numbers,

$$\cos\left[\frac{\pi(n+N)^2}{8}\right] = \cos\left[\frac{\pi(n)^2}{8}\right] \Rightarrow \frac{\pi(n+N)^2}{8} = \frac{\pi(n)^2}{8} \pm 2k\pi \Rightarrow (n+N)^2 = (n)^2 \pm 16k$$

$$\Rightarrow n^2 + N^2 + 2nN = n^2 \pm 16k \Rightarrow N^2 + 2nN = 16k \Rightarrow N = 8l$$

Answer: Therefore, the output $y[n]$ is periodic with period of $N = 8l$ and the fundamental period and frequency are $N = 8$ and $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{8} = \frac{\pi}{4}$, respectively.

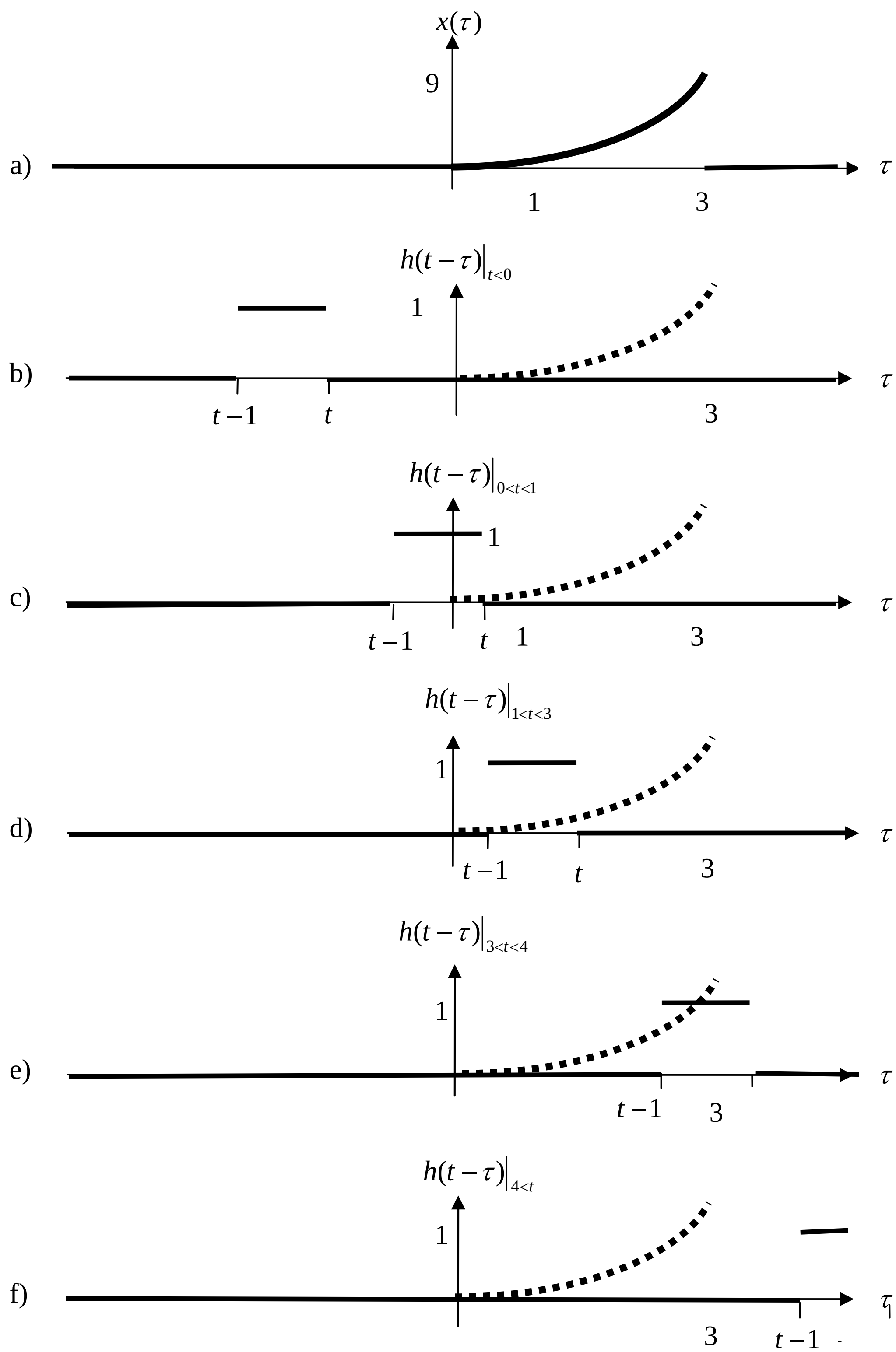
If N is an odd number N(N+2n) = (odd number) × (odd number) = odd number ≠ 16 k ※
So N is an even number: N=2M
$N^2+2nN=N(N+2n)=2M(2M+2n) = 2^2M(M+n) = 16k=2^4k \Rightarrow M(M+n) = 2^2k : \forall n\in\mathbb{Z}$
As (M+n) can take all integer values the other term, M, must have 2^2 factor, so $M = 2^2 l \Rightarrow N=2^3M=8l$

Question 3:

Consider a continuous-time LTI system which has impulse response of $h(t) = \{u(t) - u(t-1)\}$. If $x(t) = t^2\{u(t) - u(t-3)\}$ is applied at the input of the system, evaluate the output $y(t)$ of the system using convolution integral $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$ as follows:

a) Draw $x(\tau)$ and $h(t-\tau)$ for different intervals of "t".
b) Evaluate the output $y(t)$ for the intervals of "t" indicated in part (a).

Part a:


$x(\tau)$
9
a)
1
3
τ
$h(t-\tau)\big|_{t<0}$
1
b)
$t-1$
t
3
τ
$h(t-\tau)\big|_{0<t<1}$
1
c)
$t-1$
t
1
3
τ
$h(t-\tau)\big|_{1<t<3}$
1
d)
$t-1$
t
3
τ
$h(t-\tau)\big|_{3<t<4}$
1
e)
$t-1$
3
τ
$h(t-\tau)\big|_{4<t}$
1
f)
3
$t-1$
τ

Part b:

Based on the overlap of $x(\tau)$ in figure "a" and $h(t-\tau)$ in figures "b", "c", "d" "e" and "f", we can calculate the output $y(t)$ for various regions as shown below.

Answer:

$$y(t)=\begin{cases} 0 & t<0 \quad Fig.\ b) \\ \int_0^t t^2 dt = \dfrac{t^3}{3} & 0<t<1 \quad Fig.\ c) \\ \int_{t-1}^t t^2 dt = \dfrac{t^3-(t-1)^3}{3} & 1<t<3 \quad Fig.\ d) \\ \int_{t-1}^3 t^2 dt = \dfrac{27-(t-1)^3}{3} & 3<t<4 \quad Fig.\ e) \\ 0 & 4<t \quad Fig.\ f) \end{cases}$$

Question 4:

Consider two discrete-time LTI systems which are characterized by their impulse responses $h_1[n]=\delta[n]-\delta[n-1]$ and $h_2[n]=u[n]$.

a) Determine whether these two LTI systems are inverse of each other. Justify your answer.
b) Determine whether these systems are stable, memory-less, and causal. Justify your answer.

$\delta[k]=\begin{cases}0 : \forall k\neq 0\\ 1 : k=0\end{cases}$ when k≠0 then no matter what is the coefficients of delta function and when k=0 the coefficient of delta function is u[n]

or $\delta[n]*u[n]=\sum_{k=-\infty}^{k=+\infty}\delta[k]u[n-k]=$ $\cdots\delta[-1]u[n+1]+\delta[0]u[n]+\delta[1]u[n-1]+\delta[2]u[n-2]+\cdots$ $=\delta[0]u[n]=u[n]$

Part a:

$$h_1[n]*h_2[n]=\sum_{k=-\infty}^{+\infty}h_1[k]h_2[n-k]=\sum_{k=-\infty}^{+\infty}\{\delta[k]-\delta[k-1]\}u[n-k]$$

$$=\sum_{k=-\infty}^{+\infty}\delta[k]u[n-k]-\sum_{k=-\infty}^{+\infty}\delta[k-1]u[n-k]=\sum_{k=-\infty}^{+\infty}\delta[k]u[n]-\sum_{k=-\infty}^{+\infty}\delta[k-1]u[n-1]$$

Similarly
$\delta[n-1]*u[n]=\sum_{k=-\infty}^{k=+\infty}\delta[k-1]u[n-k]=\cdots+\delta[-1]u[n]+\delta[0]u[n-1]+\delta[1]u[n-2]+\cdots=$ u[n − 1]
In general we the convolution of shifted impulse and a function is the same function but shifted, *i.e.*
$\delta[n-M]*f[n]=f[n-M]$

$$=u[n]\sum_{k=-\infty}^{+\infty}\delta[k]-u[n-1]\sum_{k=-\infty}^{+\infty}\delta[k-1]=u[n]-u[n-1]=\delta[n]$$

Answer: Therefore, $h_1[n]*h_2[n]=\delta[n]$ and two systems are inverse of each other.

Part b:

System represented by $h_1[n]=\delta[n]-\delta[n-1]$ is stable, has memory and is causal.

Stability: $\sum_{k=-\infty}^{\infty}|h_1[k]|=\sum_{k=-\infty}^{\infty}|\delta[k]-\delta[k-1]|=2<\infty$ and system is stable.

Memory-less: $h_1[n]=\delta[n]-\delta[n-1]$ is not in the form of $K\delta[n]$ and has memory.

Causality: For $n<0$, $h_1[n]=\delta[n]-\delta[n-1]$ is zero and therefore it is causal.

System represented by $h_2[n] = u[n]$ is unstable, has memory and is causal.

Stability: $\sum_{k=-\infty}^{\infty} |h_2[k]| = \sum_{k=-\infty}^{\infty} |u[k]| = \sum_{k=0}^{\infty} 1 = \infty$ and system is unstable.

Memory-less: $h_2[n] = u[n]$ is not in the form of $K\delta[n]$ and has memory.

Causality: For $n < 0$, $h_2[n] = u[n]$ is zero and therefore it is causal.

Question 5:

Consider a continuous-time LTI system which has impulse response of $h(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & otherwise \end{cases}$. The input of $x(t) = \sum_{k=-\infty}^{+\infty} \delta(t-4k)$ is applied to this system.

a) Find the output. Draw both input $x(t)$ and output $y(t)$.
   Hint: Both input and output are periodic functions with fundamental period of 4.
b) Evaluate Fourier series coefficients of input $x(t)$ and output $y(t)$.

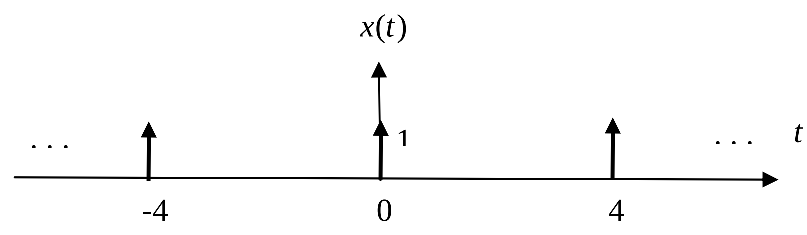
Part a:

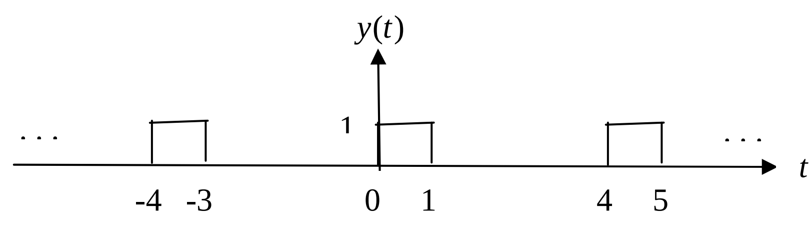
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{+\infty} \delta(\tau-4k) \right\} h(t-\tau)d\tau =$$

$$y(t) = \int_{-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{+\infty} \delta(\tau-4k)h(t-\tau) \right\} d\tau = \int_{-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{+\infty} \delta(\tau-4k)h(t-4k) \right\} d\tau =$$

$$y(t) = \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(\tau-4k)h(t-4k)d\tau = \sum_{k=-\infty}^{+\infty} \left\{ h(t-4k) \int_{-\infty}^{+\infty} \delta(\tau-4k)d\tau \right\} = \sum_{k=-\infty}^{+\infty} h(t-4k)$$

Answer: $y(t) = \sum_{k=-\infty}^{+\infty} h(t-4k)$





Part b:

Fourier series coefficients of $x(t)=\sum_{k=-\infty}^{+\infty}\delta(t-4k)$ is $a_k=\frac{1}{4}$ according to table 4.2 which can also be evaluated as follows:

$$a_k=\frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt=\frac{1}{4}\int_{-2}^{2}\delta(t)e^{-jk(\frac{2\pi}{4})t}dt=\frac{1}{4}\int_{-2}^{2}\delta(t)dt=\frac{1}{4}$$

Answer: $a_k=\frac{1}{4}$ for any integer k.

Fourier series coefficients of $y(t)$ denoted by b_k can be found as:

$$b_k=\frac{1}{T}\int_T y(t)e^{-jk(2\pi/T)t}dt=\frac{1}{4}\int_{-2}^{2}h(t)e^{-jk(\frac{2\pi}{4})t}dt=\frac{1}{4}\int_{0}^{1}e^{-jk(\frac{2\pi}{4})t}dt=\frac{1}{4}\left[\frac{e^{-jk\left(\frac{\pi}{2}\right)t}}{-jk\left(\frac{\pi}{2}\right)}\right]_0^1$$

$$b_k==\frac{1}{4}\left[\frac{e^{-jk\left(\frac{\pi}{2}\right)}}{-jk\left(\frac{\pi}{2}\right)}-\frac{1}{-jk\left(\frac{\pi}{2}\right)}\right]=\frac{1}{4}\left[\frac{1-e^{-jk\left(\frac{\pi}{2}\right)}}{jk\left(\frac{\pi}{2}\right)}\right]$$

Answer: $b_k=\frac{1-e^{-j\frac{k\pi}{2}}}{j2k\pi}$

We can also find b_k using $b_k=H(jk\omega_0)a_k$ as follows:

Fourier Transform of $h(t)$ can be found using tables 4.1 and 4.2 as follows:

$$s(t)=\begin{cases}1 & -\frac{1}{2}<t<\frac{1}{2}\\ 0 & otherwise\end{cases}\xleftrightarrow{FT^{-1}}S(j\omega)=\frac{2\sin\frac{\omega}{2}}{\omega}$$

$$h(t)=s(t-\frac{1}{2})\xleftrightarrow{FT^{-1}}H(j\omega)=e^{-j\frac{\omega}{2}}S(j\omega)=e^{-j\frac{\omega}{2}}\frac{2\sin\frac{\omega}{2}}{\omega}$$

Therefore, $H(j\omega)=\frac{2e^{-j\frac{\omega}{2}}\sin\frac{\omega}{2}}{\omega}$

Fourier series coefficient of output $y(t)$ is $b_k=H(jk\omega_0)a_k=\frac{2e^{-j\frac{k\omega_0}{2}}\sin\frac{k\omega_0}{2}}{k\omega_0}\times\frac{1}{4}$

The fundamental frequency is $\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$ and therefore: $b_k = \frac{2e^{-j\frac{k\pi}{4}} \sin\frac{k\pi}{4}}{\frac{k\pi}{2}} \times \frac{1}{4}$

Answer: $b_k = \frac{e^{-j\frac{k\pi}{4}} \sin\frac{k\pi}{4}}{k\pi} = \frac{1-e^{-j\frac{k\pi}{2}}}{j2k\pi}$ for any integer k.

Question 6:

Use the tables of properties of Fourier transforms and basic Fourier transform pairs and find:

a) $x[n]$ which is inverse Fourier transform of $X\left(e^{j\omega}\right) = \frac{e^{-j2\left(\omega-\frac{\pi}{4}\right)}}{2-e^{-j\left(\omega-\frac{\pi}{4}\right)}}$

b) Fourier transform of $y[n] = nx^*[n-3]$ where "*" means complex conjugate.

Part a:

From Tables 5.1 and 5.2:

$$\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{FT} \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$\left(\frac{1}{2}\right)^{n-2} u[n-2] \xleftrightarrow{FT} \frac{e^{-j2\omega}}{1-\frac{1}{2}e^{-j\omega}}$$

$$\frac{1}{2}\times\left(\frac{1}{2}\right)^{n-2} u[n-2] \xleftrightarrow{FT} \frac{1}{2}\frac{e^{-j2\omega}}{1-\frac{1}{2}e^{-j\omega}} = \frac{e^{-j2\omega}}{2-e^{-j\omega}}$$

$$\frac{e^{j\frac{\pi}{4}n}}{2}\times\left(\frac{1}{2}\right)^{n-2} u[n-2] \xleftrightarrow{FT} \frac{e^{-j2\left(\omega-\frac{\pi}{4}\right)}}{2-e^{-j\left(\omega-\frac{\pi}{4}\right)}}$$

$$x[n] \xleftrightarrow{FT} X\left(e^{j\omega}\right)$$

Answer: $x[n] = \frac{e^{j\frac{\pi}{4}n}}{2}\times\left(\frac{1}{2}\right)^{n-2} u[n-2]$

Part b:

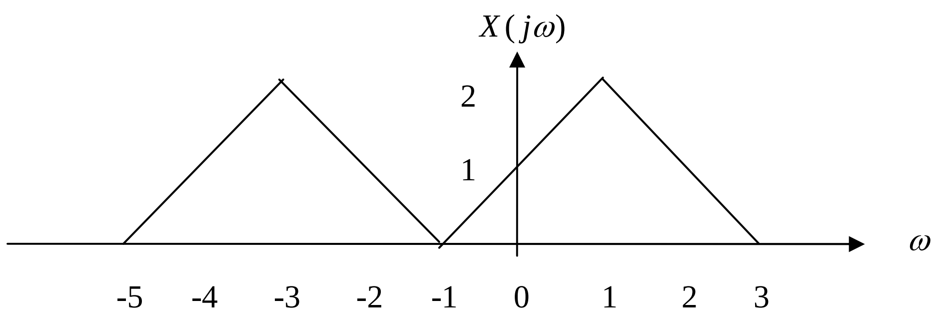
$$x[n] \xleftrightarrow{FT} X\left(e^{j\omega}\right)$$
$$x^*[n] \xleftrightarrow{FT} X^*\left(e^{-j\omega}\right)$$
$$x^*[n-3] \xleftrightarrow{FT} e^{-j3\omega} X^*\left(e^{-j\omega}\right)$$
$$nx^*[n-3] \xleftrightarrow{FT} j\frac{d}{d\omega}\left\{e^{-j3\omega} X^*\left(e^{-j\omega}\right)\right\}$$
$$y[n] = nx^*[n-3] \xleftrightarrow{FT} j\frac{d}{d\omega}\frac{e^{-j3\omega}e^{j2\left(-\omega-\frac{\pi}{4}\right)}}{2-e^{j\left(-\omega-\frac{\pi}{4}\right)}} = Y\left(e^{j\omega}\right)$$

Answer: $Y\left(e^{j\omega}\right) = j\dfrac{d}{d\omega}\dfrac{e^{-j3\omega}e^{j2\left(-\omega-\frac{\pi}{4}\right)}}{2-e^{j\left(-\omega-\frac{\pi}{4}\right)}}$

Question 7:

Fourier transform of $x(t)$ is shown in the figure as $X(j\omega)$. Without explicitly computing $x(t)$,

a) compute quantities of $\int_{-\infty}^{\infty} x(t)dt$ and $\int_{-\infty}^{\infty} |x(t)|^2 dt$,
b) compute quantities of $x(0)$ and $Phase\{x(t)\}$.



Part a:

Using defining formula for Fourier Transform to find $\int_{-\infty}^{\infty} x(t)dt$ as follows:

$$\int_{-\infty}^{\infty} x(t)dt = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\Big|_{\omega=0} = X(j\omega)\Big|_{\omega=0} = X(j0) = 1$$

Answer: $\int_{-\infty}^{\infty} x(t)dt = 1$

Use Parserval's Relation in table 4.1 to find $\int_{-\infty}^{\infty}|x(t)|^2\,dt$ as follows:

$$\int_{-\infty}^{\infty}|x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|X(j\omega)\right|^2 d\omega =$$
$$= \frac{1}{2\pi}\left[\int_{-5}^{-3}(\omega+5)^2 d\omega + \int_{-3}^{-1}(-\omega-1)^2 d\omega + \int_{-1}^{1}(\omega+1)^2 d\omega + \int_{1}^{3}(-\omega+3)^2 d\omega\right] = \frac{16}{3\pi}$$

Answer: $\int_{-\infty}^{\infty}|x(t)|^2\,dt = \frac{16}{3\pi}$

Part b:

Using basic relation of inverse Fourier Transform to find $x(0)$as follows:

$$x(0) = x(t)\big|_{t=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega\Big|_{t=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)d\omega =$$
$$= \frac{1}{2\pi}\left[\int_{-5}^{-3}(\omega+5)d\omega + \int_{-3}^{-1}(-\omega-1)d\omega + \int_{-1}^{1}(\omega+1)d\omega + \int_{1}^{3}(-\omega+3)d\omega\right] = \frac{4}{\pi}$$

Answer: $x(0) = \frac{4}{\pi}$

$Phase\{x(t)\}$is found as follows:

$X(j\omega)$ is a real and even function shifted by 1 to the left, i.e. $X(j\omega) = X_e((j(\omega+1))$ and therefore based on Frequency Shifting property of table 4.1, we have:

$$x(t) = x_e(t)e^{j(-1)t}$$

Since $X_e(j\omega)$is real and even, according to table 4.1 so is $x_e(t)$. Therefore,

$$x(t) = |x_e(t)|e^{j(-1)t}$$

Answer: From above equality $Phase\{x(t)\} = -t$

Question 8:

A discrete LTI system is described by the following difference equation:

$-6y[n] - 5y[n-1] - y[n-2] = x[n-1]$

a) Determine the frequency response and the impulse response of the system.
b) Draw the block diagram representation of the above LTI system.

Part a:

Using tables 5.1 and 5.2 we have:

$$-6y[n]-5y[n-1]-y[n-2]=x[n-1]\xleftrightarrow{FT}-6Y\left(e^{j\omega}\right)-5e^{-j\omega}Y\left(e^{j\omega}\right)-e^{-j2\omega}Y\left(e^{j\omega}\right)=e^{-j\omega}X\left(e^{j\omega}\right)$$

Therefore the frequency response of the system is,

Answer: $H\left(e^{j\omega}\right)=\dfrac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}=-\dfrac{e^{-j\omega}}{6+5e^{-j\omega}+e^{-j2\omega}}$

The impulse response of the system is $h[n]\xleftrightarrow{FT}H\left(e^{j\omega}\right)=-\dfrac{e^{-j\omega}}{6+5e^{-j\omega}+e^{-j2\omega}}$

We use partial fractions,

$$h[n]\xleftrightarrow{FT}H\left(e^{j\omega}\right)=\frac{-e^{-j\omega}}{6+5e^{-j\omega}+e^{-j2\omega}}=\frac{A}{(3+e^{-j\omega})}+\frac{B}{(2+e^{-j\omega})}=\frac{(2A+3B)+(A+B)e^{-j\omega}}{(3+e^{-j\omega})(2+e^{-j\omega})}$$

$$\begin{cases}2A+3B=0\\ A+B=-1\end{cases}\Rightarrow\begin{cases}A=-3\\ B=2\end{cases}$$

$$H\left(e^{j\omega}\right)=\frac{-3}{(3+e^{-j\omega})}+\frac{2}{(2+e^{-j\omega})}=\frac{-1}{(1+\frac{1}{3}e^{-j\omega})}+\frac{1}{(1+\frac{1}{2}e^{-j\omega})}\xleftrightarrow{FT^{-1}}h[n]=-\left(\frac{1}{3}\right)^n u[n]+\left(\frac{1}{2}\right)^n u[n]$$

Answer: $h[n]=-\left(-\dfrac{1}{3}\right)^n u[n]+\left(-\dfrac{1}{2}\right)^n u[n]$

Part b:

$$-6y[n]-5y[n-1]-y[n-2]=x[n-1]\Rightarrow\quad y[n]=-\frac{5}{6}y[n-1]-\frac{1}{6}y[n-2]-\frac{1}{6}x[n-1]$$

Answer: