Short Answer Questions:

Question 1. The domain of the function $f(x) = \frac{\ln(3x+6)}{x-2}$ is:

> **Solution:**
> $$x \neq 2 \text{ and } x > -2$$

Question 2. solve for x in $e^{\frac{1}{2}\ln(x^4)} + \ln(x) + \ln(x+2) - \ln(x^2+2x) = 9$.

> **Solution:**
> $$\begin{aligned} & e^{\frac{1}{2}\ln(x^4)} + \ln(x) + \ln(x+2) - \ln(x^2+2x) \\ =& x^2 + \ln(x(x+2)) - \ln(x^2+2x) \\ =& x^2 = 9 \end{aligned}$$
> So $x = \pm 3$.

Question 3. If $\cos(\theta) = \frac{2}{5}$ and $\frac{3\pi}{2} \leq x \leq 2\pi$, then $\tan(\theta) =$?

> **Solution:** Since $\frac{3\pi}{2} \leq x \leq 2\pi$, $\tan(\theta) < 0$. Then
> $$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = -\frac{\sqrt{1-\cos^2(\theta)}}{\cos(\theta)} = -\frac{\sqrt{1-(\frac{2}{5})^2}}{\frac{2}{5}} = -\frac{\sqrt{21}}{2}$$

Question 4. $\arccos\left(\frac{1}{\sqrt{2}}\right) =$

> **Solution:**
> $$\arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

Question 5. $\lim_{x\to\infty} \frac{\sqrt{x^2-4}}{2x+1} =$?

> **Solution:**
> $$\begin{aligned} & \lim_{x\to\infty} \frac{\sqrt{x^2-4}}{2x+1} \\ =& \lim_{x\to\infty} \sqrt{\frac{x^2-4}{(2x+1)^2}} \\ =& \sqrt{\lim_{x\to\infty} \frac{x^2-4}{(2x+1)^2}} \\ =& \sqrt{\lim_{x\to\infty} \frac{x^2-4}{4x^2+4x+1}} \\ =& \sqrt{\frac{1}{4}} = \frac{1}{2}. \end{aligned}$$

Question 6. $\lim_{x\to 3} \frac{x^3-4x^2+9}{2x-6}$ =?

Solution:

$$\lim_{x\to 3} \frac{x^3-4x^2+9}{2x-6}$$
$$=\lim_{x\to 3} \frac{3x^2-8x}{2}$$
$$=\frac{3*3^2-8*3}{2}$$
$$=\frac{3}{2}$$

Question 7. $\lim_{x\to 0} \frac{1-\cos(2x)}{\sin(3x)}$ =?

Solution:

$$\lim_{x\to 0} \frac{1-\cos(2x)}{\sin(3x)}$$
$$=\lim_{x\to 0} \frac{2\sin(2x)}{3\cos(3x)}$$
$$=\frac{2\sin(2*0)}{3\cos(3*0)}$$
$$=0$$

Question 8. $\frac{d}{dx}3^{x^2+1}$ =?

Solution:

$$\frac{d}{dx}3^{x^2+1} =3^{x^2+1}\ln(3)(x^2+1)'$$
$$=3^{x^2+1}\ln(3)(2x)$$
$$=2x\ln(3)3^{x^2+1}$$

Question 9. $\lim_{x\to 1^+}(x-1)\ln(x-1)$ =?

Solution:

$$\lim_{x\to 1^+}(x-1)\ln(x-1)$$
$$=\lim_{x\to 1^+}\frac{\ln(x-1)}{\frac{1}{x-1}}$$
$$=\lim_{x\to 1^+}\frac{\frac{1}{x-1}}{\frac{1}{(x-1)^2}}$$
$$=\lim_{x\to 1^+}(x-1)$$
$$=0$$

Question 10. The absolute minimum value of $f(x) = x^3 - 6x + 1$ on $[-1, 3]$ is

> **Solution:** First
> $$f'(x) = 3x^2 - 6$$
> Then, $f'(x) = 0$ at $x = \pm\sqrt{2}$. Compare (two end points) $f(-1) = 6$, $f(3) = 10$ and (two critical points) $f(\sqrt{2}) = 1 - 4\sqrt{2}$, $f(-\sqrt{2}) = 1 + 4\sqrt{2}$, we can see the absolute minimum value of $f(x)$ is $1 - 4\sqrt{2}$ at $x = \sqrt{2}$.

Question 11. The general antiderivative of $f(x) = 4e^{2x} + \sqrt{x} - \sin(x)$ is

> **Solution:**
> $$\begin{aligned} &\int [4e^{2x} + \sqrt{x} - \sin(x)]dx \\ =&4\int e^{2x}dx + \int x^{\frac{1}{2}}dx - \int \sin(x)dx \\ =&4 * \frac{1}{2}e^{2x} + \frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} + \cos(x) + C \\ =&2e^{2x} + \frac{2}{3}x^{\frac{3}{2}} + \cos(x) + C \end{aligned}$$

Question 12. $\frac{d}{dx}\int_0^{x^2} e^t \sin(t)dt =?$

> **Solution:**
> $$\frac{d}{dx}\int_0^{x^2} e^t \sin(t)dt = e^{x^2}\sin(x^2) * (x^2)' = e^{x^2}\sin(x^2) * (2x) = 2xe^{x^2}\sin(x^2)$$

Question 13. $\int_0^{\frac{\pi}{4}} \frac{1}{1+x^2}dx$ is

> **Solution:**
> $$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{1}{1+x^2}dx &= \arctan(x)\,\Big|_0^{\frac{\pi}{4}} \\ &= \arctan(\frac{\pi}{4}) - \arctan(0) \\ &= \arctan(\frac{\pi}{4}) \end{aligned}$$

Question 14. The inverse of $f(x) = (x-1)^3 + 4$ is

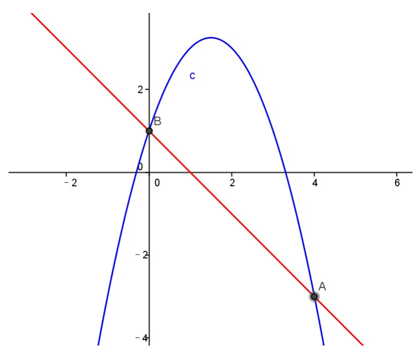
> **Solution:**
> $$y = \sqrt[3]{x-4} + 1$$

Question 15. $\int [3x - x^3 + e^x + 3\sin(x)]dx$ is

Solution:

$$\begin{aligned} &\int [3x - x^3 + e^x + 3\sin(x)]dx \\ =&3\int x dx - \int x^3 dx + \int e^x dx + 3\int \sin(x)dx \\ =&3\frac{1}{1+1}x^{1+1} - \frac{1}{3+1}x^{3+1} + e^x - 3\cos(x) + C \\ =&\frac{3}{2}x^2 - \frac{1}{4}x^4 + e^x - 3\cos(x) + C \end{aligned}$$

Question 16. Find the area of the regions enclosed by the curves of $f(x) = -x^2 + 3x + 1$ and $g(x) = -x + 1$:



Solution: First , solve $-x^2 + 3x + 1 = -x + 1$. We have $x_1 = 0$, $x_2 = 4$. So the area is

$$\int_0^4 [(-x^2 + 3x + 1) - (-x + 1)]dx = \frac{32}{3}$$

Long Answer Questions:

Question 1. (1) Determine the values of c and k that make the following function continuous.(Justify all reasoning)

$$f(x) = \begin{cases} 3x + k, & x < 0 \\ x^2 - 1 & 0 \le x \le 2 \\ \sqrt{cx+3}, & x > 2 \end{cases}$$

Solution: $f(x)$ is continuous at $x = 0$ means $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$, so

$$\lim_{x\to 0^-} f(x) = 3 * 0 + k = 0^2 - 1 = \lim_{x\to 0^+} f(x)$$

Then $k = -1$.

$f(x)$ is continuous at $x = 2$ means $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$, so

$$\lim_{x\to 2^-} f(x) = 2^2 - 1 = \sqrt{c * 2 + 3} = \lim_{x\to 2^+} f(x)$$

Then $c = 3$.

(2) Find the following limit $\lim_{x\to 0}\left(1+\frac{3}{x}\right)^{2x}$.

Solution:

$$\begin{aligned} &\lim_{x\to 0} \ln\left(1+\frac{3}{x}\right)^{2x} \\ =&\lim_{x\to 0} 2x\ln\left(1+\frac{3}{x}\right) \\ =&\lim_{x\to 0}\frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{2x}} \\ =&\lim_{x\to 0}\frac{\frac{1}{1+\frac{3}{x}} * \left(-\frac{3}{x^2}\right)}{\left(-\frac{1}{2x^2}\right)} \\ =&\lim_{x\to 0}\frac{6x}{x+3} = 0 \end{aligned}$$

So

$$\lim_{x\to 0}\left(1+\frac{3}{x}\right)^{2x} = e^0 = 1$$

(3) Find the following limit $\lim_{x\to\infty}[\sqrt{2x+1} - \sqrt{2x+7}]$.

Solution:

$$\begin{aligned} &\lim_{x\to\infty}[\sqrt{2x+1} - \sqrt{2x+7}] \\ =&\lim_{x\to\infty}\frac{[\sqrt{2x+1} - \sqrt{2x+7}][\sqrt{2x+1} + \sqrt{2x+7}]}{\sqrt{2x+1} + \sqrt{2x+7}} \\ =&\lim_{x\to\infty}\frac{(2x+1) - (2x+7)}{\sqrt{2x+1} + \sqrt{2x+7}} \\ =&\lim_{x\to\infty}\frac{-6}{\sqrt{2x+1} + \sqrt{2x+7}} \\ =&\,0 \end{aligned}$$

Question 2. (1) Find the derivatives of the following three functions:

$$f(x) = \ln(x)\sin(x), \quad g(x) = \frac{2x+1}{x^2-1}, \quad h(x) = \arctan(3x+1)$$

Solution:

$$\frac{df}{dx} = \frac{1}{x}\sin(x) + \ln(x)\cos(x)$$

$$\frac{dg}{dx} = \frac{2(x^2-1) - (2x+1)(2x)}{(x^2-1)^2} = \frac{-2x^2-2x-2}{(x^2-1)^2}$$

$$\frac{dh}{dx} = \frac{3}{1+(3x+1)^2}$$

(2) Find the equation of the tangent line of $e^y - x^2y = 2x$ at $(1, 0)$.

Solution: Differentiate both sides of the equation with respect to x, treating $y = f(x)$ as a differentiable function of x, then we have

$$e^y y' - (2xy + x^2y') = 2$$

Move all terms involving y' to the left side of the equation and move all other terms to the right side of the equation.

$$(e^y - x^2)y' = 2 + 2xy$$

then

$$y' = \frac{2+2xy}{e^y - x^2}.$$

So the slope of the tangent line is $\frac{2+2xy}{e^y-x^2}\Big|_{x=1,y=0} = \frac{2}{0} = \infty$, Thus, the tangent line is $x = 1$.

(3) Determine the linearization of $g(x) = \sqrt[3]{x}$ at $x = 27$ and use this equation to approximate the value of $\sqrt[3]{26}$.

Solution: Since $g'(x) = \frac{1}{3}x^{-2/3}$, we have

$$g(27) = \sqrt[3]{27} = 3$$

$$g'(1) = \frac{1}{3}x^{-2/3}|_{x=27} = \frac{1}{27}$$

The linearization is

$$L(x) = 3 + \frac{1}{27}(x - 27)$$

and

$$\sqrt[3]{26} \approx 3 + \frac{1}{27}(26-27) = 3 - \frac{1}{27}$$

(4) Using logarithmic differentiation, find the derivative of $f(x) = \frac{e^{2x}(x^3-1)^3}{x^5(1-3x)^6}$.

Solution: Apply ln to both sides and use laws of logarithms:

$$\ln f(x) = 2x + 3\ln(x^3-1) - 5\ln x - 6\ln(1-3x)$$

Take the derivative of both sides of this equality:

$$\frac{d}{dx}[\ln f(x)] = \frac{d}{dx}\left[2x + 3\ln(x^3-1) - 5\ln x - 6\ln(1-3x)\right]$$
$$\frac{f'}{f} = 2 + \frac{9x^2}{x^3-1} - \frac{5}{x} - \frac{-18}{1-3x}$$

Multiply both sides by f to get f'

$$f' = f\left[2 + \frac{9x^2}{x^3-1} - \frac{5}{x} + \frac{18}{1-3x}\right]$$
$$= \left[\frac{e^{2x}(x^3-1)^3}{x^5(1-3x)^6}\right]\left[2 + \frac{9x^2}{x^3-1} - \frac{5}{x} + \frac{18}{1-3x}\right]$$

Question 3. (1) Consider a funciton $f(x) = 12\sqrt[3]{x^2}e^{\frac{1}{6}x}$ which has the derivative given as:

$$f'(x) = 2x^{-\frac{1}{3}}e^{\frac{1}{6}x}(4+x)$$

i. Determine the intervals on which f is increasing or decreasing.

ii. Find all critical points and classify the points as local max, local min or neither.

Solution: $f'(x) = 0$ at $x = -4$ and $x = 0$. These two critical points subdivide the domain of f to three non-overlapping open intervals:

Interval	$-\infty < x < -4$	$-4 < x < 0$	$0 < x < \infty$
Sign of f'	+	−	+
Behavior of f	Increasing	Decreasing	Increasing

We can see from the table that there is a local maximum $f(-4) = 12 * 2^{\frac{4}{3}}e^{-\frac{2}{3}}$ at $x = -4$ and a local minimum $f(0) = 0$ at $x = 0$.

(2) Consider a function $g(x) = \frac{x^2-1}{2x^2-8}$ that has the following properties:

- $D = (-\infty,\ -2) \cup (-2,\ 2) \cup (2,\ \infty)$.
- Intercepts: $(0,\ 0.125)$, $(-1,\ 0)$ and $(1,\ 0)$.
- Critical points: $x = 0$.
- g is increasing on $(-\infty,\ -2)$ and $(-2,\ 0)$; decreasing on $(0,\ 2)$ and $(2,\ \infty)$.
- g is concave up on $(-\infty,\ -2)$ and $(2,\ \infty)$; concave down on $(-2,\ 2)$.
- Vertical asymptotes: $x = 2$ and $x = -2$; horizontal asymptotes: $y = 0.5$.

Provide a sketch of the function.

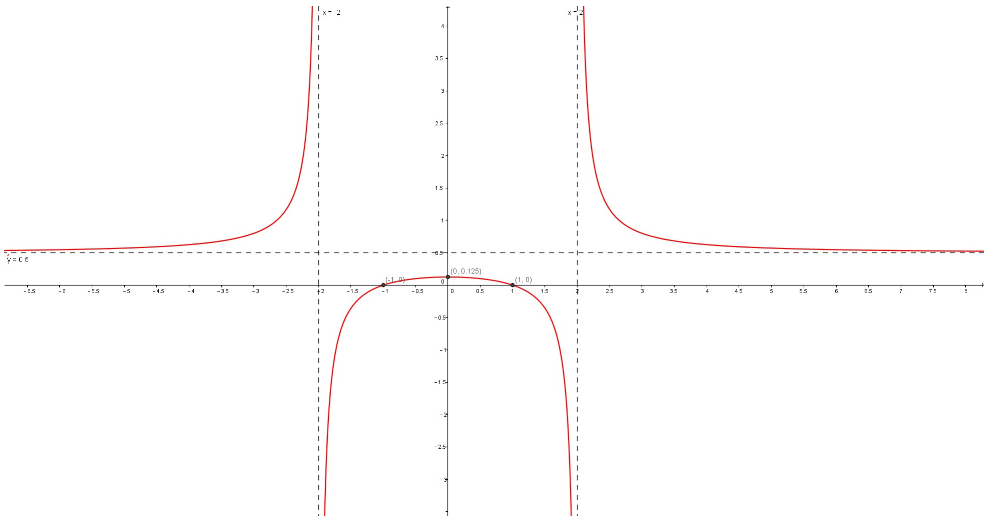


Figure 1: $g(x) = \frac{x^2-1}{2x^2-8}$

Question 4. Given $f(x) = x^2 - 1$ and $g(x) = 2x + 7$,

(1) Determine the points of intersection of the two functions.

Solution: solve $x^2 - 1 = 2x + 7$. We have $x_1 = -2$, $x_2 = 4$, so the points of intersection of these two functions are $(-2,\ 3)$, $(4,\ 15)$

(2) Determine the contained area between the two curves.

Solution:

$$\begin{aligned}
\text{Area } &= \int_{-2}^{4} [(2x+7) - (x^2-1)]dx \\
&= \int_{-2}^{4} [8 + 2x - x^2]dx \\
&= [8x + x^2 - \frac{x^3}{3}] \,|_{-2}^{4} \\
&= [8 * 4 + 4^2 - \frac{4^3}{3}] - [8(-2) + (-2)^2 - \frac{(-2)^3}{3}] \\
&= \frac{108}{3}
\end{aligned}$$

(3) Let $H(x)$ be the antiderivative of $f(x)$, determine the function $H(x)$ given that the point $(-3,\ 1)$ lies on the graph of $H(x)$.

Solution: First

$$H(x) = \int f(x)dx = \int (x^2-1)dx = \frac{x^3}{3} - x + C.$$

Since $(-3,\ 1)$ lies on the graph of $H(x)$, then

$$1 = \frac{(-3)^3}{3} - (-3) + C$$

So $C = 7$. Thus

$$H(x) = \frac{x^3}{3} - x + 7.$$