

• **6.4 Area and Riemann Integral**

Let  $[a, b]$  be a closed interval of real numbers i.e.,  $[a, b] = \{x : a \leq x \leq b\}$ . A **partition  $\mathcal{P}$  of  $[a, b]$**  means breaking the interval by a finite number of points  $a = x_0, x_1, x_2, \dots, x_n = b$  into a finite number of smaller intervals, where these points are arranged in ascending order as

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

The length of each subinterval of the form  $[x_i, x_{i+1}] = x_{i+1} - x_i = \Delta x_i$

**Norm:**  $|\mathcal{P}|$  = length of the largest subinterval

Let  $f$  be a function with domain  $[a, b]$ , the Riemann sum to approximate the area under the curve for a given partition  $\mathcal{P}$  of an interval  $[a, b]$  is given by

$$\sum_{i=0}^{n-1} f(t_i) \Delta x_i = f(t_0)(x_1 - x_0) + f(t_1)(x_2 - x_1) + \dots + f(t_{n-1})(x_n - x_{n-1})$$

Limit of the Riemann sum as the norm of the partition  $\mathcal{P}$  approaches to 0 is  $L$

$$\lim_{|\mathcal{P}| \rightarrow 0} \sum_{i=0}^{n-1} f(t_i) \Delta x_i = L$$

We say that  $f$  is **Riemann Integrable** over the interval  $[a, b]$  and  $L$  is the **value of the Riemann Integral of  $f$  over  $[a, b]$** , denoted by

$$L = \mathcal{R} \int_a^b f(x) dx = \mathcal{F}(b) - \mathcal{F}(a) = \lim_{|\mathcal{P}| \rightarrow 0} \sum_{i=0}^{n-1} f(t_i) \Delta x_i$$

The symbols on the right side call  **$\mathcal{R}$ -integral** of  $f$  over  $[a, b]$ .

For  $f(x) \geq 0$ , area under  $f$  between the two lines at  $a$  and  $b$  is

$$\int_a^b f(x) dx = \mathcal{F}(b) - \mathcal{F}(a) = \mathcal{F}(x) \Big|_a^b$$

**The fundamental Theorem of Calculus:**

If  $f'$  is continuous over  $[a, b]$ , then

$$\int_a^b f'(x) dx = f(b) - f(a)$$

While  $f$  is continuous over  $[a, b]$ , then

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

and

$$\int_{a(x)}^{b(x)} f(t) dt = \mathcal{F}(b(x)) - \mathcal{F}(a(x)).$$

By using chain rule

$$\begin{aligned} \frac{d}{dx} \int_{a(x)}^{b(x)} f(t) dt &= \frac{d}{dx} (\mathcal{F}(b(x)) - \mathcal{F}(a(x))) \\ &= \mathcal{F}'(b(x))b'(x) - \mathcal{F}'(a(x))a'(x) \\ &= f(b(x))\frac{db}{dx} - f(a(x))\frac{da}{dx} \end{aligned}$$

### Examples:

1. Evaluate:

$$\begin{aligned} &\frac{d}{dx} \int_{-x^2+1}^4 \cos(t^2 \ln t) dt \\ &\frac{d}{dx} \int_{x^2}^{\sin x} 3t^2 dt \\ &\int_0^{\frac{\pi}{2}} \sin^4 x \cos x dx \\ &\int_0^2 x e^{x^2} dx \\ &\lim_{x \rightarrow 0} \frac{d}{dx} \int_0^{\sqrt{x}} \frac{\sin(y^2)}{2y} dy \end{aligned}$$

2. Show that  $\lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \frac{1}{n+i} = \ln 2$

3. Find

$$\int_0^{\frac{\pi}{6}} \frac{\sin 2x}{\cos^2 2x} dx$$

4. Find  $I'(x)$  if

$$I(x) = \int_1^{x^2} \frac{1}{1 + \sqrt{1-t}} dt$$

5. Find the area under the curve  $f(t) = t\sqrt{1-t^2}$  over interval  $[0, 1]$

6. Find the solution for the following differential equation

$$\frac{dy}{dx} = (1 + y^2)e^{2x}, \quad y(0) = 1 \qquad \frac{d^2y}{dx^2} = e^x, \quad y(0) = 0 \quad y'(0) = 0$$

## 7. Techniques of Integration

- 7.1 Trigonometric Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\sec^2 x - \tan^2 x = 1$$

$$\csc^2 x - \cot^2 x = 1$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\sin 2x = 2 \sin x \cos x$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

**Example:** Find

$$\cos(x - y) \qquad \sin(x - y)$$

- 7.2 Substitution Rule

$$\int_a^b f(x) dx = \mathcal{F}(b) - \mathcal{F}(a)$$

The substitution  $u(x) = t \rightarrow x = u^{-1}(t)$  inside the integral corresponds to the **change of variable formula**

$$\mathcal{F}(b) - \mathcal{F}(a) = \int_{u(a)}^{u(b)} f(u^{-1}(t)) \left( \frac{d}{dt} u^{-1}(t) \right) dt$$

**Example:** Evaluate

$$\begin{aligned} & \int_0^2 2x e^{x^2} dx \\ & \int_0^\pi x \sin x^2 dx \\ & \int e^y \sec e^y \tan e^y dy \\ & \int \frac{\tan(\ln x)}{x} dx \end{aligned}$$

- Hyperbolic Functions:** These functions are denoted by  $\sinh, \cosh, \dots$  which analogous to the usual circular functions or trigonometric functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

Note that

$$\int \sinh u \, du = \cosh u + C, \quad \int \cosh u \, du = \sinh u + C$$

**Examples:** Evaluate

$$\begin{aligned} & \int x^2 \sqrt{1+x^3} dx \\ & \int_0^1 \frac{x^3 dx}{\sqrt[4]{1+x^4}} \\ & \int \frac{y^2+1}{y^3+3y+1} dy \\ & \int_1^2 x \sqrt{x-1} dx \\ & \int (\ln x)^3 \frac{dx}{x} \\ & \int \frac{e^{-x}}{1+e^{-x}} dx \\ & \int_0^{\frac{1}{3}} \frac{dx}{1+9x^2} \\ & \int \frac{y+1}{y^2+2y} dy \end{aligned}$$