

Week # 7                      MATH1004  
Integration

• **6.1 Antiderivatives and Indefinite Integral**

The procedure of the finding the antiderivative of a given function is the inverse operation to that of differentiation. It means, for a given  $f$  continuous on  $[a, b]$  its antiderivative is a differentiable function  $\mathcal{F}$  such that

$$\mathcal{F}'(x) = f(x) \quad a < x < b$$

The derivative of an antiderivative of a function is the function itself.

Find an antiderivative,  $\mathcal{F}(x)$  of  $f(x) = 2x$

The derivative  $f'$  of a given function  $f$  is always unique but the antiderivative,  $\mathcal{F}$  is not unique. In other word:

an antiderivative of a function is defined up to the addition of an arbitrary constant.

$$\mathcal{F}_1 - \mathcal{F}_2 = C$$

Another notation for an antiderivative of a given function  $f$  is given by  $\int$  which is called the **indefinite integral** of  $f$ , that is

$$\begin{aligned}\mathcal{F}(x) &= \int^x f(t)dt && t \text{ is free variable, means,} \\ \mathcal{F}(x) &= \int^x f(u)du\end{aligned}$$

Note that

$$\mathcal{F}'(t) = \frac{d}{dx} \int^x f(t)dt = f(x)$$

**Power Rule**

For any  $r \neq -1$

$$\int^x t^r dt = \frac{x^{r+1}}{r+1} + C$$

In general, for a given constant  $c$

$$\int^x ct^r dt = \frac{cx^{r+1}}{r+1} + C$$

$C$  is called a **constant of integration**

An antiderivative of a sum of two or more functions is the sum of the antiderivatives of each of the functions

$$\int^x (f(t) \pm cg(t))dt = \int^x f(t)dt \pm c \int^x g(t)dt$$

**Examples:**

1. Evaluate:

$$\int (3x^2 - 2x + 1.3x^5)dx$$

2. Find antiderivative of  $f(x) = 4x^3 - 2$  where  $\mathcal{F}(0) = 1$ .

• **Generalized Power Rule**

$$\int u(x)^r \frac{du}{dx} dx = \frac{u(x)^{r+1}}{r+1} + C \quad r \neq -1$$

**Examples:**

1. Evaluate

a.  $\int (2x^3 + 1)x^2 dx$     b.  $\int \sin^2 x \cos x dx$     c.  $\int e^{5x} dx$

2. Solve  $\frac{dy}{dx} = \frac{x}{y}$ ,  $y > 0$

3. Solve  $\frac{d^3 y}{dx^3} = 6$  given that  $y = 5$ ,  $\frac{dy}{dx} = 0$ ,  $\frac{d^2 y}{dx^2} = -8$ , when  $x = 0$

4. Evaluate

a.  $\int x\sqrt{4x^2 + 1} dx$

b.  $\int \cos^2 x \sin x dx$     given that  $\mathcal{F}(0) = 0$

• **6.2 Definite Integrals**

Let  $[a, b]$  be a given interval,  $f$  a given function with domain  $[a, b]$ , assumed **continuous** on its domain. The **definite integral of f over  $[a, b]$**  is, by definition, an expression of the form

$$\int_a^b f(x)dx = \mathcal{F}(b) - \mathcal{F}(a) \quad \text{definite integral over } [a, b]$$

where  $\mathcal{F} = f$ , ie.,  $\mathcal{F}$  is antiderivative of  $f$ .

$$\int_0^1 (2x + 1)dx$$

• **Properties of the Definite Integral:**

$$\int_a^a f(x) = 0$$

$$\int_a^b f(x)dx = - \int_b^a f(x)dx$$

$$\int_a^b kf(x)dx = k \int_a^b f(x)dx \quad k \text{ is a constant} \quad \int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\text{If } f(x) \geq 0 \text{ and } a \leq b \rightarrow \int_a^b f(x)dx \geq 0$$

$$\text{If } f(x) \leq g(x), x \in [a, b] \rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

**Example:** Evaluate

$$\int_0^1 \frac{1}{1+x^2} dx$$

$$\int_0^1 xe^{-x^2} dx$$

$$\int_0^\pi \cos^2 x \sin x dx$$

$$\int_0^1 3^x dx$$

$$\int_0^{0.5} \frac{x}{\sqrt{1-x^2}} dx$$

$$\int_0^1 x2^{x^2+1} dx$$

$$\int_{-1}^1 \frac{x}{1+x^4} dx$$

• **6.3 The Summation Convention**

The summation sign  $\sum$  is defined by the symbol

$$\sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$$

for example  $\sum_{i=1}^5 i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$

We can show that

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

Let  $a_i, b_i$  are finite sequences of numbers

$$\begin{aligned}\sum_{i=m}^n (a_i \pm b_i) &= \sum_{i=m}^n a_i \pm \sum_{i=m}^n b_i \\ \sum_{i=m}^n c a_i &= c \sum_{i=m}^n a_i \\ \sum_{i=m}^n c &= c(n - m + 1)\end{aligned}$$

**Examples:**

1. Write in  $\sum$  form

a.  $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1$

b.  $\frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \cdots + \frac{n}{n}$

2. Expand and find the value of

a.  $\sum_{k=1}^6 \frac{(-1)^k}{k^2}$

b.  $\sum_{i=1}^{100} i^2$

c.  $\sum_{i=1}^n \frac{i}{n}$