

Week # 6 MATH1004
Curve Sketching

• **5.1 Solving Polynomial Inequalities:**

Step I: Factoring the polynomial $p(x)$ in two types

1. Type I (or linear) factor

$$a_1x - a_2$$

2. Type II (or Quadratic Irreducible) factor

$$a^2x + bx + c \text{ where } b^2 - 4ac < 0$$

Examples: Factors into a product of Type I and/or Type II factors

$$x^4 - 1, \quad x^6 - 1, \quad x^3 - x^2 - x + 1, \quad x^4 - 16$$

Step II: The Sign Decomposition Table (SDT) of Polynomial

Break points: The points x where $f(x) = 0$

The SDT of $f(x) = x^4 - 1$

	$x + 1$	$x - 1$	$x^2 + 1$	sign of $f(x)$
$(-\infty, -1)$	-	-	+	+
$(-1, 1)$	+	-	+	-
$(1, \infty)$	+	+	+	+

The number of rows=(The total number of different break points)+1

The number of columns=(The total number of different break points + total number of different type II factors)+1

Example: Find the SDT of the polynomial

$$(x^4 - 1)(3 + \cos x)$$

Step III: Read off the Solution of Inequality

Examples: Solve the following inequalities

$$p(x) = (x - \frac{1}{2})(x + 2.6)(x - 1)^2(x^2 + x + 1) < 0 \qquad p(x) = 3(x^2 - 4)(9 - x^2) \leq 0$$

• 5.2 Solving Rational Function Inequalities

A rational function is the quotient of two polynomials.

A **break point** of a rational function r is any real root of either its numerator or its denominator but not a root of both.

See the difference for

$$r(x) = \frac{x^2 - 9}{x - 3}, \quad r(x) = \frac{(x^2 - 9)^2}{x - 3}.$$

Examples: Find break points:

$$r(x) = \frac{x^2 + 1}{x}, \quad r(t) = \frac{3 - t^2}{t^3 + 1}, \quad r(t) = x + 1 + \frac{2}{x - 1}, \quad \frac{x^2 - 2x + 1}{x^2 - 1}$$

The SDT of a rational function is found in exactly the same way as the SDT for a polynomial. The only difference is that we have to include all the break-points of the numerator and denominator which make it up, without common roots.

Example: Solve the following inequality:

$$\begin{aligned} \frac{(x - 2)(x + 4)}{x^2 - 9} &< 0 \\ \frac{x}{3x - 6} + \frac{2x}{x - 2} &< 0 \\ \frac{4 - t^2}{1 - t^2} &\geq 0 \end{aligned}$$

• 5.3 Graphing Techniques

Answer the following questions for sketching the graph of a function

1. What are the *zeros* of f and the y - intercepts?
2. What are the critical points of f , and what is their nature?
3. Where is f increasing/decreasing?
4. Where is the graph of f concave or/concave down?
5. What are the points of inflection?
6. Where are the asymptotes of f , if any, and identify them, (horizontal or vertical).
7. Put all information together and sketch the graph.

1. zeros of f and the y - intercepts:

The value of y when $x = 0$ and value/s of x when $y = 0$.

2. Critical Points: A point c in the domain of a differentiable function f is called “critical point” if either

1. $f'(c) = 0$
2. $f'(c)$ does not exist

Note: Endpoint is critical since two-sided derivative does not exist.

Example: Find critical points for

$$f(x) = |x|, \quad g(x) = x^{\frac{2}{3}}, \quad h(x) = x^2$$

3. Increasing and Decreasing Functions: A function f defined on an interval, I , is said to be “increasing” if give any pair of points x_1, x_2 , in I with $x_1 < x_2 \rightarrow f(x_1) < f(x_2)$. It is decreasing if $x_1 < x_2 \rightarrow f(x_1) > f(x_2)$.

Let f be differentiable over (a, b)

- i If $f'(x) > 0$ for all x in (a, b) then f is increasing on $[a, b]$
- ii If $f'(x) < 0$ for all x in (a, b) then f is decreasing on $[a, b]$

Example: Determine the intervals on which the function f is increasing and decreasing.

$$f(x) = x + \frac{5}{6} \ln |x - 3| - \frac{5}{6} \ln |x + 3|$$

Local Maximum: $x = a$ is a local maximum if in the neighborhood $f(x) < f(a)$.

Local Minimum: $x = a$ is a local minimum if in the neighborhood $f(x) > f(a)$.

Global Maximum: $x = a$ is a global maximum if $f(x) < f(a)$ for every x in the domain of f .

Global Minimum: $x = a$ is a global minimum if $f(x) > f(a)$ for every x in the domain of f .

• **First Derivative Test:**

Let $f'(c) = 0$ with f differentiable function

- i If $f'(x) > 0$ in a left-neighborhood of c and $f'(x) < 0$ in a right-neighborhood of c , then $f(c)$ is a local maximum value of f .
- ii If $f'(x) < 0$ in a left-neighborhood of c and $f'(x) > 0$ in a right-neighborhood of c , then $f(c)$ is a local minimum value of f .

4. **Concavity:** A graph of twice differentiable function is said to be “concave up” on an interval I if $f''(x) > 0$ and it is “concave down” if $f''(x) < 0$ for every x in I .

5. **Point of Inflection:** A point with coordinate $(c, f(c))$ on the graph of a twice differentiable function f if

1. $f''(x) > 0$ in a left-neighborhood of c and $f''(x) < 0$ in right neighborhood of c or
2. $f''(x) < 0$ in a left-neighborhood of c and $f''(x) > 0$ in right neighborhood of c .

- **The Second Derivative Test**

If $f''(x) < 0$, then c is local maximum of f

If $f''(x) > 0$, then c is local minimum of f

If $f''(x) = 0$, then more information is needed.

what is the point of inflection? The tangent line at an inflection point divides the small portion of the curve into two parts, one part of the curve is on one side of the tangent line, while the other part of the curve is on the other side.

Example: Determine critical points and point of inflection for $f(x) = x^3 - 3x + 1$.

- **6. Horizontal and Vertical Asymptotes**

The horizontal lines $y = L, y = M$ on the xy -plane are called “horizontal asymptotes for the graph of f ” if

$$\lim_{x \rightarrow \infty} f(x) = L \text{ or } \lim_{x \rightarrow -\infty} f(x) = M$$

The vertical line $x = a$ is called a “vertical asymptote for the graph of f ” provided either

$$\lim_{x \rightarrow a} f(x) = \infty, \text{ or } \lim_{x \rightarrow a^+} f(x) = \infty, \text{ or } \lim_{x \rightarrow a^-} f(x) = \infty \text{ (or } -\infty)$$

Example: Determine vertical and horizontal asymptotes:

$$f(x) = \frac{x}{|x| + 2}, \quad f(x) = \frac{2x^2}{x^2 + 3x - 4}$$

Check List:

- Monotonicity

- Horizontal Asymptotes
- Zeros
- Concavity
- Extrema
- Vertical Asymptotes
- Points of Inflection

Example: Sketch the graph for

$$f(x) = \frac{4}{9 + x^2}$$