

• 4.1 Exponential and Logarithms

$$f(x) = a^x \quad a = \text{base, } a > 0 \quad x = \text{power or exponent}$$

its inverse is

$$F(x) = \log_a x$$

Note:

$$x = f(F(x)) = a^{\log_a x} \quad \text{or} \quad x = F(f(x)) = \log_a a^x$$

in general:

$$\square = a^{\log_a \square} \quad \text{or} \quad \square = \log_a a^\square$$

Example: Write the following expression as logarithm:

$$2^{-\frac{1}{2}} = \frac{\sqrt{2}}{2} \quad 4^{\sqrt{2}} \approx 7.10299 \quad \sqrt[3]{3}^{\sqrt{3}} \approx 1.38646$$

$$a^x = y \quad \text{means} \quad \log_a y = x$$

• Properties:

$$\log_a x^y = y \log_a x$$

$$\log_a xy = \log_a x + \log_a y$$

$$\log_a x/y = \log_a x - \log_a y$$

$$a^0 = 1 \quad \log_a 1 = 0 \quad \log_a a = 1$$

Example:

1. Solve $\log_2 x = -3$

2. Evaluate $\log_3 \sqrt{3}$

3. Simplify $f(x) = \log_2(\sqrt{x} \sin x)$ $0 < x < \frac{\pi}{2}$ $f(x) = \log_4(4^x 8^{x^2+1})$, $x \geq 0$ $\log_a(2^x 2^{-x})$

4. Write as power function $\log_{\frac{1}{3}} 27 = -3$

5. Solve for x $\log_3\left(\frac{x}{x+1}\right) = 1$, $\log_{\sqrt{2}}(x^2 - 1) = 0$

• 4.2 Definition of Euler's Number $e = 2.718281 \dots$

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

Example: Evaluate $\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n$

In general:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^n = e^a$$

- **4.3 Euler's Exponential Function and Natural Logarithm**
Properties of e^x

$$e^{x+y} = e^x e^y$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{x-y} = e^x / e^y$$

$$\lim_{x \rightarrow \infty} e^x = \infty, \quad \lim_{x \rightarrow -\infty} e^x = 0$$

$$\frac{d}{dx} e^x = e^x$$

- Inverse of exponential function is **natural logarithm**, $\ln(x) = \log_e x$

$$e^{\ln x} = x, x > 0 \quad \ln e^x = x, \quad \ln e = 1, \quad \ln 1 = 0$$

$$\ln x^y = y \ln x, \quad \ln xy = \ln x + \ln y, \quad \ln x/y = \ln x - \ln y$$

$$e^{\ln \square} = \square \quad \text{if} \quad \square = a^x \quad \text{then} \quad a^x = e^{x \ln a}$$

- **Change of Base**

$$\log_a x = \frac{\ln x}{\ln a} \quad \log_a x = \frac{\log_b x}{\log_b a} \quad (\text{in general})$$

Rewrite by using e :

$$(2.3)^{1.2}$$

$$f(x) = 2^{\sin x}$$

$$g(x) = x^x$$

- **4.4 Derivative of the Natural Logarithm**

$$\frac{d}{dx} \ln x = \frac{1}{x}, \quad \frac{d}{dx} \ln \square = \frac{1}{\square} \frac{d}{dx}(\square), \quad \square > 0$$

Examples Find derivative:

$$1. \ln(x^2 + 2) \text{ at } x = 0$$

$$2. e^x \ln x$$

$$3. \ln(x^2 + 2x + 1)$$

$$4. 3 \ln(x + 1)$$

$$5. e^{2x} \ln(x^2 + 1)$$

$$6. \frac{e^x}{\ln x}$$

$$7. \ln \sqrt{x^2 + 4}$$

$$8. \ln(x + \sqrt{x^2 + 3})$$

• **4.5 Differentiation Formula for General Exponential Functions**

$$\frac{d}{dx} a^{h(x)} = a^{h(x)} \ln a h'(x)$$

Examples: Find derivative:

1. $e^{-1.6x}$
2. $2^{\sin x}$
3. $(e^x)^{-2}$
4. $(1.3)^{x^2} \cos x$
5. $x^2 e^{2x}$
6. $3^{\cos x}$

• **4.6 Differentiation Formula for General Logarithm Functions**

$$\frac{d}{dx} \log_a h(x) = \frac{1}{\ln a} \frac{1}{h(x)} \frac{d}{dx} h(x)$$

Examples: Find derivative:

1. $\log_4(2x + 1)$ at $x = 0$
2. $2^x \log_3 3x$
3. $3^x \log_2(x^2 + 1)$
4. $\log_2 \sqrt{x + 1}$