

- **Inverse functions**

Let f be a given function with $D = \text{Dom}(f)$ and $R = \text{Ran}(f)$, we say that function F is the inverse of f if

- $\text{Dom}(F) = \text{Ran}(f)$, $\text{Dom}(f) = \text{Ran}(F)$
- $(F \circ f)(x) = x$ for every x in $\text{Dom}(f)$

The inverse function of f is usually written f^{-1} whereas the reciprocal function of f is written as $\frac{1}{f}$.

- How to find the inverse of a function

1. Write $y = f(x)$ and solve for x in terms of y , then $x = F(y)$ is the inverse.
2. Interchange the x 's and y 's that gives $y = f(x)$.

The graph of inverse function, F , is obtained by reflecting the graph of f about the line $y = x$. **Example** Find inverse function for $f(x) = x^3 + 1$

- **Derivatives of inverse function**

$$\frac{dF}{dx}(x) = \frac{df^{-1}}{dx}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(F(x))}$$

Examples:

1. Evaluate $\frac{df^{-1}}{dx}(16)$ if $f(x) = x^4$, $x \geq 0$.
2. Given that f is such that its inverse F exists. $f'(-2.1) = 4$, $F(-1) = -2.1$, find the value of the derivative of F at $x = -1$.

3.

$$f(x) = \frac{2+3x}{3-2x}, \quad x \neq \frac{3}{2}, \quad f^{-1}(x) = ?, \quad \text{Dom}(f) = ?, \quad \text{Ran}(f) = ?$$

4.

$$g(t) = \sqrt{1-4t^2}, \quad 0 \leq t \leq \frac{1}{2}, \quad g^{-1}(t) = ?, \quad \text{Dom}(g) = ?, \quad \text{Ran}(g) = ?$$

- **Inverse Trigonometric Functions**

Inverse of $\sin x$ defined when $\text{Dom}(\sin) = (-\frac{\pi}{2}, \frac{\pi}{2})$ and called $\text{Arcsin}(x)$
 $y = \text{Arcsin}(x)$ means that y is an angle whose sin is x

Function	Domain	Range
$y = \text{Arc sin}(x) = \sin^{-1}(x)$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \text{Arc cos}(x) = \cos^{-1}(x)$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \text{Arc tan}(x) = \tan^{-1}(x)$	$-\infty \leq x \leq \infty$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \text{Arc cot}(x) = \cot^{-1}(x)$	$-\infty \leq x \leq \infty$	$0 \leq y \leq \pi$

Example: Find

$$\text{Arc sin}(\frac{1}{2}), \quad \sin(\text{Arc cos}(\frac{\sqrt{2}}{2}))$$

$$\cos(\text{Arc sin}(0)), \quad \text{Arc sin}(\tan(-\frac{\pi}{4}))$$

• Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx} \text{Arc sin } x = ?$$

$$\begin{aligned} \frac{d}{dx} \sin^{-1} u &= \frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \\ \frac{d}{dx} \cos^{-1} u &= \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx} \\ \frac{d}{dx} \tan^{-1} u &= \frac{1}{1+u^2} \frac{du}{dx} \\ \frac{d}{dx} \cot^{-1} u &= \frac{-1}{1+u^2} \frac{du}{dx} \end{aligned}$$

Examples:

1. Find y' if $y = \cos^{-1}(\frac{1}{x})$
2. Find $\frac{dy}{dx}$ if $y = \cot^{-1}(\sqrt{x})$
3. Find derivative of $x^2 \text{Arc cos } x$

• L'Hospital's Rule

1. The rule is about limit.
2. The rule always involves a **QUOTIENT** of two functions.

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \quad \text{indeterminate form}$$

RULE:

Let f and g be two functions defined and differentiable in a punctured neighborhood

of a , when a is finite. If $g'(x) \neq 0$ in this punctured neighborhood of a , and $\frac{f(x)}{g(x)}$ is one of the $\frac{\infty}{\infty}$ or $\frac{0}{0}$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

The rule also hold if a is replaced by $\pm\infty$ or if the limits are one sided limit.

1. $\lim_{x \rightarrow 0} \frac{\sin x}{x}$
2. $\lim_{x \rightarrow 0} \frac{\tan(\alpha x)}{x}$
3. $\lim_{x \rightarrow 0} \frac{\sqrt{x+1}-1}{x^2}$
4. $\lim_{x \rightarrow 0} \frac{\tan 2x - 2x}{x - \sin x}$
5. $\lim_{x \rightarrow 3} \frac{x^2 - 4x + 3}{x^2 + x - 12}$
6. $\lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 7x}$