

• **Derivatives**

Tangent to a curve :

$$y - y_0 = m(x - x_0) \quad \text{where}$$

$$m = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{derivative of } f \text{ at } x_0$$

• **Notation for Derivatives:**

$$f'(x_0), \frac{df}{dx}(x_0), D_x f(x_0), Df(x_0)$$

The instantaneous rate of change of f at $x = x_0$

Note: If the limit as $h \rightarrow 0$ does not exist as a two-sided limit or it is infinite we say that the **derivative does not exist**. This is equivalent to saying that there is no uniquely defined tangent line at $(x_0, f(x_0))$.

$$\left\{ \begin{array}{ll} f'(x) = 0 & \text{horizontal tangent line;} \\ f'(x) > 0 & \text{tangent line rise;} \\ f'(x) < 0 & \text{tangent line falls;} \\ f'(x) & \text{sometimes does not exist.} \end{array} \right.$$

Examples:

1. Find the slope of tangent at $x = 2$ for $y = \frac{1}{x}$.
2. Find derivatives of $f(x) = |x|$ at $x = 0$.

In general:

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

The slope of the tangent line at $x = x_0$

The instantaneous rate of change of f at $x = x_0$.

whenever two-sided limit exists and is finite.

Examples:

1. Find

$$\lim_{h \rightarrow 0} \frac{f(2 + h) - f(2)}{h} \quad \text{where } f(x) = \sqrt{x}$$

2. Find

$$\lim_{h \rightarrow 0^-} \frac{f(1 + h) - f(1)}{h}, \quad \lim_{h \rightarrow 0^+} \frac{f(1 + h) - f(1)}{h}, \quad \text{where}$$

$$f(x) = \begin{cases} x+1 & x \geq 1 \\ x & 0 \leq x < 1 \end{cases}$$

3. find the slope of $f(x) = x|x|$ at $x = 0$.

4. If

$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ x+2 & 1 \leq x < 2 \\ 8-x^2 & 2 \leq x < 3 \end{cases}$$

what is $f'(x)$? Does $f'(2)$ exist? What is $f'(\frac{5}{2})$?

When a given function f has a derivative at $x = a$, f is differentiable at $x = a$.

- **The Power Rule**

$$\frac{d}{dx}x^n = nx^{n-1}$$

- **Sum/Difference Rule**

$$\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx} = f'(x) \pm g'(x)$$

- **Product Rule**

$$\frac{d}{dx}(fg) = f'(x)g(x) + g'(x)f(x)$$

- **Quotient Rule**

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Examples:

1. Find $f'(x)$ at $x = -1$, if $f(x) = \sqrt[3]{x} + \sqrt[3]{2} - 1$
2. At which points on graph of $y = x^3 + 3x$ does the tangent line have slope equal to 9.
3. Find $f'(x)$ if

$$f(x) = \frac{x^{\frac{2}{3}}}{\sqrt{x} + 3x^{\frac{3}{4}}}$$

- **The Chain Rule**

Let f , g , be two differentiable functions with g differentiable at x and $g(x)$ in the domain of f' . Then composition of two functions $f \circ g$ is differentiable at x and .

$$\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

By using box method

$$\frac{d}{dx}f(\square) = f'(\square)\square'$$

Examples:

1. Find $y'(0)$ if $y(x) = x^2 - \frac{6}{x-4}$
2. Find $f'(x)$ if $f(x) = \sqrt{\sqrt{x} + 1}$

- **Consequences of the Chain Rule**

$$\frac{d}{dx} \left(\frac{1}{g(x)} \right) = \frac{-1}{(g(x))^2} g'(x) \quad \text{Reciprocal Rule}$$

$$\frac{d}{dx} (g(x))^a = a(g(x))^{a-1} g'(x) \quad \text{Generalized Power Rule}$$

Example Find $f'(x)$ if $f(x) = (x^{2/3} + 1)^2$

- **Higher Order Derivatives**

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \frac{dy}{dx} = y''(x)$$

Examples:

1. Find $f''(x)$ if $f(x) = \frac{1}{1+\sqrt{x}}$.
2. Given $y(x) = f(g(x))$, $g'(2) = 1$, $g(2) = 0$, $f'(0) = 1$, what is the value of $y'(2)$?
3. Let $f(x) = \sqrt{x + \sqrt{x}}$, evaluate $f'(9)$. If $x = t^2$, what is $\frac{df}{dt}$?

- **Implicit Functions and their Derivatives**

An equation involving two variables, say x , y is said to be an **explicit relation** if one can solve for y uniquely in terms of x .

An equation involving two variables, say x , y is said to be an **implicit relation** if it is not explicit.

Examples:

1. Find $\frac{dy}{dx}$ if $xy - y^2 = 6$
2. Find $\frac{dx}{dy}$ at the point $(x, y) = (-3, 1)$, if $y^5 + x^2 y^3 = 10$.
3. Find tangent line to $(x + y)^3 - x^3 - y^3 = 0$ at $(-1, 1)$.

- **Derivatives of Trigonometric Functions**

There are two fundamental limits for sin and cos

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

• **Fundamental trigonometric identities**

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\sin^2 x + \cos^2 x = 1$$

$$\sec^2 x - \tan^2 x = 1$$

$$\csc^2 x - \cot^2 x = 1$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\sin 2x = 2 \sin x \cos x$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

• **Derivatives for Trigonometric Functions**

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx} \cot x = \frac{-1}{\sin^2 x} = -\csc^2 x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

Examples: 1. Find $f'(x)$ if $f(x) = \sin^2 x + 6x$

2. Find $\frac{d}{dx} \left(\frac{\cos x}{1 + \sin x} \right)$

3. Find $h'(t)$ at $t = \frac{\pi}{4}$, if $h(t) = \frac{t}{\sin 2t}$

4. Find derivative of $(\sin 3x)^{-1}$

5. Find derivative of $\cos(x \sin x)$

6. If

$$y(x) = \begin{cases} \frac{\sin x}{\tan x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

a. Show y is continuous at $x = 0$.

b. Show y is differentiable at $x = 0$

c. Conclude that $y'(0) = 0$