

- **One-sided Limit of a Functions**

**Limits from the right:** The function  $f$  has a limit from the right at  $x = a$  whose value is  $L$  and denote by

$$f(a+0) = \lim_{x \rightarrow a^+} f(x) = L$$

if **BOTH** of the following statements are satisfied:

1. Let  $x > a$  and  $x$  be very close to  $x = a$
2. As  $x$  approaches  $a$  (from the right), the value of  $f(x)$  approaches to the value  $L$ .

**Example:**

$$F(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

$$\lim_{x \rightarrow 0^+} F(x) = 1$$

**Limits from the left:** The function  $f$  has a limit from the left at  $x = a$  whose value is  $L$  and denote by

$$f(a-0) = \lim_{x \rightarrow a^-} f(x) = L$$

if **BOTH** of the following statements are satisfied:

1. Let  $x < a$  and  $x$  be very close to  $x = a$
2. As  $x$  approaches  $a$  (from the left), the value of  $f(x)$  approaches to the value  $L$ .

$$\lim_{x \rightarrow 0^-} F(x) = 0$$

**Examples:**

1. Find

$$\lim_{x \rightarrow 2^+} \left( \frac{x-2}{x+2} \right), \quad \lim_{x \rightarrow 0^+} (x|x|)$$

2. If

$$g(x) = \begin{cases} x^2 + 1 & x < 0 \\ 1 - x^2 & 0 \leq x \leq 1 \\ x & x > 1 \end{cases}$$

find  $\lim_{x \rightarrow 0^-} g(x)$ ,  $\lim_{x \rightarrow 0^+} g(x)$ ,  $\lim_{x \rightarrow 1^-} g(x)$ ,  $\lim_{x \rightarrow 1^+} g(x)$

Conclude that the graph of  $g$  has no break at  $x = 0$ , but it does have at  $x = 1$

- **Two-sided Limits:**  $f$  has the two-sided limit  $L$  as  $x$  approaches  $a$  if

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$$

then

$$\lim_{x \rightarrow a} f(x) = L \quad \text{the limit of } f(x) \text{ as } x \text{ approaches to } a \text{ is } L.$$

- **Continuity:**  $f$  is continuous at  $x = a$  if

1.  $f$  is defined at  $x = a$
2.  $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$
3.  $L = f(a)$

It is easier to remember

$$f \text{ is continuous at } x = a \text{ if } \lim_{x \rightarrow a} f(x) = f(a)$$

**Example:** Show  $f$  is continuous at  $x = 1, x = 2$  if

$$f(x) = \begin{cases} x + 1 & 0 \leq x \leq 1 \\ 2x & 1 < x \leq 2 \\ x^2 & x > 2 \end{cases}$$

- **Properties of Limits of Functions:**

Let  $f, g$  be two given functions and  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  both exist and are finite, then

$$\begin{aligned} \lim_{x \rightarrow a} (f(x) \pm g(x)) &= \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) \\ \lim_{x \rightarrow a} cf(x) &= c \lim_{x \rightarrow a} f(x) \\ \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0 \\ \lim_{x \rightarrow a} f(x)g(x) &= \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x) \\ f(x) \leq g(x) &\rightarrow \lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x) \end{aligned}$$

**Examples:**

1. Determine whether the following limits exist.

$$\begin{aligned} \lim_{x \rightarrow 0} f(x) &= \begin{cases} x + 2 & x \leq 0 \\ x & x > 0 \end{cases} \\ \lim_{x \rightarrow 0} \left(\frac{2}{x}\right) \\ \lim_{x \rightarrow 2} (2 + |x - 2|) \end{aligned}$$

2. Is  $f(x) = \frac{x^2+1}{x^2-2}$  continuous at 0?

- **Some continuous functions:**

Let  $x = a$  be a given point

a. The Polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

is continuous at any real number  $x = a$ .

b. The Rational function

$$r(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0}$$

is continuous at  $x = a$  if  $q(a) \neq 0$

c. If  $f$  is a continuous function, so its absolute value function  $|f|$  and

$$\lim_{x \rightarrow a} |f(x)| = 0 \rightarrow \lim_{x \rightarrow a} f(x) = 0$$

**Example:** Determine discontinuity points for the following functions

$$g(x) = \begin{cases} x & x < 0 \\ 1 + x^2 & x \geq 0 \end{cases}$$

$$f(x) = \frac{x^2 + 3x + 3}{x^2 - 1}$$

- **Trigonometry**

Angles in Radian:      Radian =  $\frac{\text{degree} \times \pi}{180}$

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\cos(-\theta) = \cos \theta, \text{ (even function)} \quad \sin(-\theta) = -\sin \theta \text{ (odd function)}$$

- Continuity of trigonometric functions:

1.  $\sin x$ ,  $\cos x$  are continuous everywhere.
2.  $\tan x$ ,  $\sec x$  are continuous except at  $(\text{odd} \times \frac{\pi}{2})$
3.  $\csc x$ ,  $\cot x$  are continuous except at  $(\text{any number} \times \pi)$

- Important limits:

$$\lim_{x \rightarrow 0} \sin x = 0, \quad \lim_{x \rightarrow 0} \cos x = 1, \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

**Examples:** Find

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 2x}{3x} \\ \lim_{x \rightarrow 0} \frac{1 - \cos 3x}{4x} \\ \lim_{x \rightarrow \pi} \frac{\sin x}{x - \pi} \\ \lim_{x \rightarrow \frac{\pi}{2}} \left(x - \frac{\pi}{2}\right) \tan x \\ \lim_{x \rightarrow 0} \frac{\cos x - \cos 2x}{x^2} \end{aligned}$$

- **How to find**  $\lim_{x \rightarrow a} f(x)$ ?

*Option 1:* Take the value to which  $x$  tends,  $x = a$  and evaluate the function at that value,  $f(a)$ . There are 3 possibilities:

- Obtain a number like  $\frac{A}{B}$  with  $A \neq 0$  which is the answer.
- Obtain  $\frac{B}{0}$  with  $B \neq 0$  which answer is  $+\infty$  or  $-\infty$ .
- Obtain  $\frac{0}{0}$ , to find answer, move to option 2.

*Option 2:* try to simplify the expression, then repeat option 1.

*Option 3.* If options 1 and 2 fail, check left and right limits. The limit exists if they are equal, otherwise limit does not exist.

- **Intermediate Value Theorem**(Bolzano's Theorem)

Let  $f$  be continuous at each point of a close interval  $[a, b]$ , assume

- $f(a) \neq f(b)$
- $z$  be a point between  $f(a)$  and  $f(b)$

Then there is at least one value of  $c$  between  $a$  and  $b$  such that  $f(c) = z$

- **Evaluating Limits at Infinity**

For any real number  $c$

$$\lim_{x \rightarrow \infty} \frac{c}{x^r} = 0, \quad r > 0$$

- **Sandwich Theorem:**

If

$$g(x) < f(x) < h(x)$$

for all large  $x$  and real number  $A$

$$\lim_{x \rightarrow \infty} g(x) = A, \quad \lim_{x \rightarrow \infty} h(x) = A$$

then  $f$  has a limit in infinity and

$$\lim_{x \rightarrow \infty} f(x) = A$$

**Examples:** Find

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{3x^2 - 2x + 1}{x^2 + 2} \\ \lim_{x \rightarrow \infty} \sqrt{x^2 + x + 1} - x \\ \lim_{x \rightarrow \infty} \sqrt{x}(\sqrt{x+1} - \sqrt{x}) \\ \lim_{x \rightarrow -\infty} \frac{\cos x}{x^2} \end{aligned}$$

- **How to Guess a Limit**

Table of value and look for pattern.

**Note:** The most common **indeterminate** forms are

$$0, (\pm\infty), \quad \pm \frac{\infty}{\infty}, \quad \infty - \infty, \quad (\pm\infty)^0, \quad 1^{\pm\infty}, \quad \frac{0}{0}, \quad 0^0$$

**Examples:** 1. If

$$g(x) = \begin{cases} x^2 + 1 & x < 0 \\ 1 - |x| & 0 \leq x \leq 1 \\ x & x > 1 \end{cases}$$

a) Evaluate

$$\lim_{x \rightarrow 0^-} g(x), \quad \lim_{x \rightarrow 0^+} g(x), \quad \lim_{x \rightarrow 1^-} g(x), \quad \lim_{x \rightarrow 1^+} g(x)$$

b) Show  $g$  has not breaks at  $x = 0$ , but it has a break at  $x = 1$

2. Whether  $\lim_{x \rightarrow 1} f(x)$  exist if

$$f(x) = \begin{cases} \frac{\sin(x-1)}{x-1} & 0 \leq x < 1 \\ 1 & x = 1 \\ |x - 1| & x > 1 \end{cases}$$

3. Find

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{bx}{\sin(ax)} \\ \lim_{x \rightarrow \infty} \frac{\cos(3x)}{4x} \end{aligned}$$

4. Use Bolzano's theorem and calculator to prove that the function  $f(x) = x \sin x + \cos x$  has a root in the interval  $[-5, 1]$

- **Limits equal to  $\pm\infty$**

The following limits exist but are equal to either  $+\infty$  or  $-\infty$ .

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1}, \quad \lim_{x \rightarrow \infty} x^2, \quad \lim_{x \rightarrow -\infty} x$$

but limit does not exist for some functions like

$$\lim_{x \rightarrow \infty} \sin(x)$$

**Examples:** Find

$$\lim_{x \rightarrow \infty} \frac{\sin 2x}{\sin 4x}, \quad \lim_{x \rightarrow -\infty} \frac{x^4}{x^3 + 1}$$