

Week # 11 MATH1004
7.8 Improper Integrals

- In some cases a definite integral may have one or both of its limits infinite, this integral is an example of an **Improper Integral**.

Such integrals can be grouped into two basic classes.

1. Those with one or both limits being infinite.
2. Those with both limits being finite, but the integrand being infinite somewhere inside the interval of integration.

Case one: Improper integral with infinite limit(s).

$$\int_a^\infty f(x)dx = \lim_{T \rightarrow \infty} \int_a^T f(x)dx, \quad \int_{-\infty}^a f(x)dx = \lim_{T \rightarrow -\infty} \int_T^a f(x)dx$$

1. The improper integral converges if the limit exists and is finite
2. The improper integral diverges if the limit does not exist at all.
3. the improper integral converges to $\pm\infty$ if the limit exists but its value is $\pm\infty$.

Examples:

1. Determine the value of p for which the improper integral

$$\int_1^\infty \frac{1}{x^p} dx$$

converges or diverges.

2. Evaluate

$$\int_0^\infty x e^{-x} dx, \quad \int_0^\infty x^2 e^{-x} dx, \quad \int_0^\infty e^{-x} \sin x dx$$

Case two: Improper integral with an infinite discontinuity

Let f has discontinuous over $[a, b)$ and assume that f has an **infinite discontinuity at $x=b$** . We define the improper integral of f over $[a, b)$ by

$$\int_a^b f(x)dx = \lim_{T \rightarrow b^-} \int_a^T f(x)dx$$

which is a limit from the left at $x = b$.

Let f has discontinuous over $(a, b]$ and assume that f has an **infinite discontinuity at $x=a$** . We define the improper integral of f over $(a, b]$ by

$$\int_a^b f(x)dx = \lim_{T \rightarrow a^+} \int_T^b f(x)dx$$

is a limit from the right at $x = a$.

Let f be continuous over (a, b) and assume that f has an **infinite discontinuity at both $x=a$ and $x=b$** . Let c be any point inside (a, b) , then We define the improper integral of f over (a, b) by

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

where the first integral is defined by a limit from the right and the second integral is defined by a limit from the left as defined above.

Example: Evaluate

$$\int_0^1 \frac{1}{\sqrt{x}} dx, \quad \int_0^2 \frac{1}{x^2 - 4x + 3} dx$$

8.2 Finding the Area Between Two Curves

A method for finding the area of an arbitrary closed region in two dimensional space is described. The method of slicing is used.

Step 1. Draw the region and find all the points of intersection of the curves.

Step 2. Choose a vertical or horizontal slice.

vertical slice area=(height)(width)=(difference in the y -coordinates)(dx)

Horizontal slice area=(width)(height)=(difference in x -difference)(dy)

Examples:

1. The region R is bounded by the curve $y = e^x$ and $y = 4 \sin x$, $x = 0.5$, and $x = 1$. Find the area of typical vertical slice and area of typical horizontal slice.
2. A region in the xy -plane has a vertical slice with coordinates $(x, x^2 e^x)$ and (x, e^{2x}) with $0 \leq x \leq 1$. What is its area?
3. A vertical slice of the closed region bounded by the curves $y = x^2 - 1$ and $y = 0$

between $x = 0$ and $x = 1$.

4. A horizontal slice of closed region bounded by the curves $y = x^2 - 1$ and $y = 0$ between $x = 0$ and $x = 1$.

Step 3. If the width is dx , find the left-most point ($x = a$) and the right most point ($x = b$). If the height is dy , find the bottom-most point ($y = c$) and the top-most point ($y = d$).

Step 4. Set up the definite integral

$$\int_a^b (\text{height of a typical slice})dx \quad \text{or} \quad \int_c^d (\text{width of a typical slice})dy$$

Examples:

1. Find the area of the region R enclosed by the curves $y = x^2$, $y = 4 - 3x^2$ and the vertical lines $x = -1$ and $x = 1$.
2. Find the area enclosed by the triangle bounded by lines $y = x$, $y = 4 - x$ and x -axis.
3. Find the expression for the area of the region bounded by the curves $y = 2x - x^2$, $y = x^3 - x^2 - 6x$ and between lines $x = -1$ and $x = 0$.
4. Find the area of the region bounded by the curves $y = x$, $y = x^3$ between lines $x = 0$ and $x = 1$.
5. Find the area of the region bounded on the right by $y = 6 - x$ and left by $y = \sqrt{x}$ and below by $y = 1$.
6. Find the area of the region bounded by the curves $y = x^2 - 1$ and $y = 3$
7. Find the area of the region bounded by the curve $y = x^2 + 5x + 6$, $y = e^{2x}$ and $x = 0$.