

## 7.4 Partial Fraction

### • 7.4.1 Long Division of Polynomials

$$\frac{p(x)}{q(x)} \quad \text{such that } \deg p(x) \geq \deg q(x)$$

In general

$$\frac{p(x)}{q(x)} = (\text{new polynomial}) + \frac{\text{remainder}}{\text{denominator}}$$

**Example:** Simplify

$$\frac{x^4 + 3x^2 - 2x + 1}{x + 1}, \quad \frac{3x^4 - 8x^3 + 20x^2 - 11x + 8}{x^2 - 2x + 5}$$

### • 7.4.2 The Integration of Partial Fraction

$$\frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}$$

- If  $n \geq m$ : Use long division to have a polynomial and remainder

- If  $n < m$ :

1. Factor denominator completely to product of linear and irreducible factors and their powers

2. If the denominator has a linear factor  $(x - r)^p$  then partial fraction decomposition is

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} \cdots + \frac{A_p}{(x - r)^p}$$

3. If the denominator has a quadratic irreducible factor of form  $(ax^2 + bx + c)^q$ , then partial fraction decomposition is form

$$\frac{B_1 x + C_1}{ax^2 + bx + c} + \frac{B_2 x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_q x + C_q}{(ax^2 + bx + c)^q}$$

- To find As, Bs and Cs:

**Method 1.** The plug in method

**Method 2.** Comparing coefficients

**Method 3.** Derivative method

**Examples:**

1. Decompose:

$$f(x) = \frac{x - 4}{x(x - 1)^2}, \quad f(x) = \frac{3x + 2}{x^4 - 1}, \quad f(x) = \frac{x^3 + 2x^2 - 1}{(x - 1)^3(x^3 - 1)}$$

2. Evaluate:

$$\int \frac{5x - 7}{x^2 - 3x + 2} dx$$

3. Evaluate

$$\int \frac{x+1}{(x-1)^2(x+2)} dx \quad \int \frac{3}{x^2(x^2+9)} dx$$
$$\int_0^1 \frac{3x^2}{x^2+2x+1} dx, \quad \int \frac{dx}{1+\sqrt{x}}, \quad \int_0^1 \ln(x^2+1) dx$$

4. Evaluate

$$\int \frac{x^4+1}{x^2+1} dx$$

## 7.5 Product of Trigonometric Functions

### • 7.5.1 Product of Sines and Cosines

$$\int \cos^2 x \sin^3 x dx, \quad \text{or} \quad \int \sin^5 x dx, \quad \text{or} \quad \int \cos^4 x dx$$

The identities

$$\cos^2 u = \frac{1 + \cos 2u}{2} \quad \sin^2 u = \frac{1 - \cos 2u}{2} \quad \sin 2u = 2 \sin u \cos u$$

are used to reduce the power in a trigonometric integral by 1

**Example:** Evaluate:

$$\int \cos^4 x dx, \quad \int \sin^4 x dx$$

In addition use the following identities for two angles:

$$\sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}, \quad \sin A \cos B = \frac{\sin(A-B) + \sin(A+B)}{2},$$

$$\cos A \cos B = \frac{\cos(A-B) + \cos(A+B)}{2}$$

To evaluate

$$\int \cos^m x \sin^n x dx$$

• **m is odd n is even:** Use  $\cos^2 x = 1 - \sin^2 x$  and then substitution of variable

**Example** Evaluate:

$$\int \cos^3 x dx, \quad \int \cos^3 x \sin^4 x dx$$

• **m is odd, n is odd:** Factor out a copy of  $\sin x$  and  $\cos x$ , then using  $\sin^2 x + \cos^2 x = 1$  for either of even power, then substitution method.

**Example:** Evaluate

$$\int \sin^3 x \cos^3 x dx$$

- **m is even n is odd:** Substitute  $\sin^2 x$  by  $1 - \cos^2 x$

**Example:** Evaluate

$$\int \sin^3 x \cos^2 x dx$$

- **m is even n is even:** Remove all power by using

$$\cos^2 x = \frac{1 + \cos 2x}{2}, \quad \sin^2 x = \frac{1 - \cos 2x}{2}$$

**Example:** Evaluate

$$\int \sin^4 x \cos^2 x dx, \quad \int x \cos^2(3x^2 + 1) dx, \quad \int x \sin^2(x^2) \cos^2(x^2) dx$$

## 7.6 Trigonometric Substitution

- **7.6.1 Completing the Square Substitution**

Making square form for irreducible polynomials

$$ax^2 + bx + c = a \left[ \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2} \right]$$

Rewrite the following polynomials by “completing the square”

$$x^2 - x + 1 \qquad 2x^2 - 4x + 4$$

Evaluate

$$\int_0^1 \frac{1}{4x^2 - 4x + 2} dx \qquad \int \frac{1}{\sqrt{2x - x^2}} dx \qquad \int \frac{1}{x^2 - x + 1} dx \qquad \int \frac{1}{\sqrt{2x - x^2 + 1}} dx$$

### 7.6.2 Trigonometric Substitution

A trigonometric substitution is used when the integrand has a particular form, eg. it can be turned into the sum or a difference of two squares, one of which is constant.

*Useful substitution for the following forms:*

$\sqrt{a^2 - u^2}$	substitute $u = a \sin \theta$ , $du = a \cos \theta d\theta$ , $\sqrt{a^2 - u^2} = a \cos \theta$
$\sqrt{a^2 + u^2}$	substitute $u = a \tan \theta$ , $du = a \sec^2 \theta d\theta$ , $\sqrt{a^2 + u^2} = a \sec \theta$
$\sqrt{u^2 - a^2}$	substitute $u = a \sec \theta$ , $du = a \sec \theta \tan \theta d\theta$ , $\sqrt{u^2 - a^2} = a \tan \theta$

**Example:** Evaluate

$$\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1 - 4x^2}}, \quad \int \frac{1}{x\sqrt{x^2 - 16}} dx, \quad \int \sqrt{2x - x^2} dx$$