MAT 2384 3X Assignment #1 Solutions

1. $y' = 6x\sqrt{1-y^2}$, $y(0)=0$

this DE is separable : $\dfrac{dy}{\sqrt{1-y^2}} = 6x\,dx$

so we integrate on both sides $\displaystyle\int \frac{dy}{\sqrt{1-y^2}} = \int 6x\,dx + C$

to get $\arcsin y = 3x^2 + C$

and so, the general solution is $y = \sin(3x^2+C)$

then $y(0)=0 \Rightarrow 0 = \sin(C) \Rightarrow C = 0$

$\therefore$ the unique solution is $\boxed{y = \sin(3x^2)}$

---

2. $(2x+1)\,dx + (4y+2)\,dy = 0$, $y(2)=1$

this DE is separable (and exact): $(4y+2)\,dy = -(2x+1)\,dx$

and so $\displaystyle\int (4y+2)\,dy = -\int (2x+1)\,dx + C$

we get $2y^2 + 2y = -x^2 - x + C$

or $2y^2 + 2y + x^2 + x = C$ (general solution)

then $y(2)=1 \Rightarrow 2(1)^2 + 2(1) + (2)^2 + (2) = C \Rightarrow C = 10$

$\therefore$ the unique solution is $\boxed{2y^2 + 2y + x^2 + x = 10}$

---

3. $(x+2y)\,dx - x\,dy = 0$, $y(1) = 5$

this DE is not separable, nor is it exact

but $\left.\begin{array}{l} M(x,y) = x + 2y \\ N(x,y) = -x \end{array}\right\}$ are both homogeneous of degree 1

so we make the substitution $y = ux$, $dy = u\,dx + x\,du$

and the DE becomes $(x+2ux)dx - x(u\,dx + x\,du) = 0$

or $x\,dx + 2ux\,dx - xu\,dx - x^2\,du = 0$

or $x\,dx + ux\,dx - x^2\,du = 0$

or $x(1+u)\,dx - x^2\,du = 0$

which is separable $\frac{1}{1+u}\,du = \frac{1}{x}\,dx$

integrate $\int \frac{du}{1+u} = \int \frac{dx}{x} + C$

which gives $\ln|1+u| = \ln|x| + C$

or $1+u = Kx$

so $u = Kx - 1$

but $u = y/x$, so $y = (Kx-1)x = Kx^2 - x$ (general solution)

then $y(1) = 5 \Rightarrow 5 = K(1)^2 - 1 \Rightarrow K = 6$

$\therefore$ the unique solution is $\boxed{y = 6x^2 - x}$

---

4. $(2x\cos y + y^2)\,dx + (2xy - x^2\sin y)\,dy = 0, \quad y(1) = \pi$ (not separable)

$M(x,y) = 2x\cos y + y^2 \Rightarrow M_y = -2x\sin y + 2y$
$N(x,y) = 2xy - x^2\sin y \Rightarrow N_x = 2y - 2x\sin y$
$\Big\}$ $M_y = N_x$ DE is exact

$F(x,y) = \int M(x,y)\,dx + g(y)$ (or $\int N(x,y)\,dy + g(x)$)
$= \int (2x\cos y + y^2)\,dx + g(y)$
$= x^2\cos y + xy^2 + g(y)$

then $\frac{\partial F}{\partial y} = -x^2\sin y + 2xy + g'(y) = N(x,y) = 2xy - x^2\sin y$

$\Rightarrow g'(y) = 0 \Rightarrow$ take $g(y) = 0$

and so $F(x,y) = x^2\cos y + xy^2$

and the general solution is $x^2\cos y + xy^2 = C$

then $y(1)=\pi \Rightarrow (1)^2\cos(\pi) + (1)(\pi)^2 = C \Rightarrow C = \pi^2 - 1$

$\therefore$ the unique solution is $\boxed{x^2\cos y + xy^2 = \pi^2 - 1}$

---

5. $(8xe^y + 3y\sin x)\,dx + (4x^2e^y - 3\cos x + 2y)\,dy = 0$, $y(0)=3$ (not sep)

$M(x,y) = 8xe^y + 3y\sin x \Rightarrow M_y = 8xe^y + 3\sin x$
$N(x,y) = 4x^2e^y - 3\cos x + 2y \Rightarrow N_x = 8xe^y + 3\sin x$
$\Big\}$ $M_y = N_x$ DE is exact

$F(x,y) = \int M(x,y)\,dx + g(y)$ (or $\int N(x,y)\,dy + g(x)$)
$= \int(8xe^y + 3y\sin x)\,dx + g(y)$
$= 4x^2e^y - 3y\cos x + g(y)$

then $\dfrac{\partial F}{\partial y} = 4x^2e^y - 3\cos x + g'(y) = N(x,y) = 4x^2e^y - 3\cos x + 2y$

so $g'(y) = 2y \Rightarrow g(y) = y^2$

and so $F(x,y) = 4x^2e^y - 3y\cos x + y^2$

and the general solution is $4x^2e^y - 3y\cos x + y^2 = C$

then $y(0)=3 \Rightarrow 4(0)^2e^3 - 3(3)\cos(0) + (3)^2 = C \Rightarrow C = 0$

$\therefore$ the unique solution is $\boxed{4x^2e^y - 3y\cos x + y^2 = 0}$

6. $f(x) = x^3 + 8x - 7$ $\quad$ $\left.\begin{array}{l} f(0) = -7 \\ f(1) = 2 \end{array}\right\}$ function changes sign on interval $\therefore$ root in $[0,1]$

$$x^3 + 8x - 7 = 0 \Rightarrow 8x = 7 - x^3 \Rightarrow x = \frac{7 - x^3}{8}$$

we can take $g(x) = \dfrac{7-x^3}{8}$

then $|g'(x)| = \left| -\frac{3}{8}x^2 \right| = \frac{3}{8}x^2 \le \frac{3}{8}$ on $[0,1]$

so this $g(x)$ will generate a convergent sequence

$x_0 = 0.75$, $\quad x_1 = g(x_0) = \dfrac{7-(0.75)^3}{8} = 0.82227$

$$x_2 = g(x_1) = \frac{7-(0.82227)^3}{8} = 0.80551$$

$$x_3 = g(x_2) = \frac{7-(0.80551)^3}{8} = 0.80967$$

$$x_4 = g(x_3) = \frac{7-(0.80967)^3}{8} = 0.80865$$

$$x_5 = g(x_4) = \frac{7-(0.80865)^3}{8} = 0.80890$$

$$x_6 = g(x_5) = \frac{7-(0.80890)^3}{8} = 0.80884$$

$$x_7 = g(x_6) = \frac{7-(0.80884)^3}{8} = 0.80885$$

$$x_8 = g(x_7) = \frac{7-(0.80885)^3}{8} = 0.80885 = x_7$$

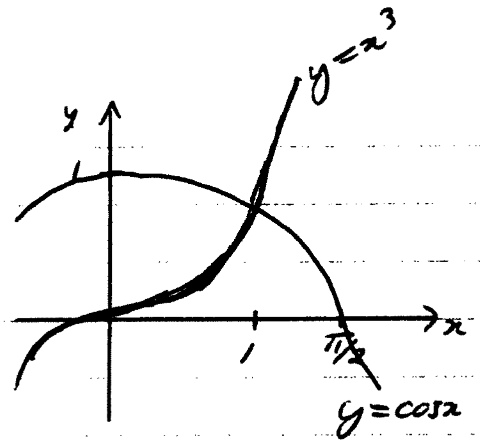
$\therefore$ stop

$\therefore$ the root is $\boxed{0.80885}$ $\quad$ (check: $f(0.80885) \approx -1.9 \times 10^{-5}$ okay)

3X15

A1 5

7.



let $f(x) = x^3 - \cos x$

then $f'(x) = 3x^2 + \sin x$

Newton's Method $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^3 - \cos(x_n)}{3x_n^2 + \sin(x_n)}$

$x_0 = 1,\quad x_1 = 1 - \dfrac{1 - \cos(1)}{3 + \sin(1)} = 0.880\,333$

$x_2 = 0.880\,333 - \dfrac{(0.880\,333)^3 - \cos(0.880\,333)}{3(0.880\,333)^2 + \sin(0.880333)} = 0.865\,684$

$x_3 = 0.865\,684 - \dfrac{(0.865\,684)^3 - \cos(0.865\,684)}{3(0.865684)^2 + \sin(0.865684)} = 0.865\,474$

$x_4 = 0.865474 - \dfrac{(0.865474)^3 - \cos(0.865474)}{3(0.865474)^2 + \sin(0.865474)} = 0.865474$

$= x_3$

$\therefore$ stop

check: $(0.865\,474)^3 = 0.648\,279$

$\cos(0.865474) = 0.648\,279$

$\therefore$ the point of intersection is $\boxed{(0.865\,474,\ 0.648\,279)}$