COMP 233, Winter 2013 **CONCORDIA UNIVERSITY**

PROBABILITY AND STATISTICS FOR COMPUTER SCIENCE.
Assignment 1. Solution.

1. (4 points) Prove for arbitrary sets A, B, C that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC).$$

Solution.

This can be done in several ways, for example,

$$\begin{aligned} P(A \cup B \cup C) &= P((A \cup B) \cup C) = P(A \cup B) + P(C) - P((A \cup B)C) \\ &= P(A) + P(B) - P(AB) + P(C) - P(AC \cup BC) \\ &= P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC) \end{aligned}$$

2. (4 points) If $P(E) = 0.9$ and $P(F) = 0.9$, show that $P(EF) \geq 0.8$. In general, prove Bonferroni's inequality, namely that

$$P(EF) \geq P(E) + P(F) - 1.$$

Solution.

$$1 \geq P(E \cup F) = P(E) + P(F) - P(EF)$$

implies Bonferroni's inequality

$$P(EF) \geq P(E) + P(F) - 1.$$

Using Bonferroni's inequality with the values provided we get

$$P(EF) \geq P(E) + P(F) - 1 = 0.9 + 0.9 - 1 = 0.8.$$

3. An urn contains four balls, labeled 1 to 4. Balls are drawn at random one by one, without replacement, until the sum of the numbers on the balls drawn exceeds 4. The sequence of balls drawn is noted. [*Note*: "exceeds 4" means "strictly larger than 4".]

   (a) (2 points) Write down the sample space for this experiment.

   (b) Let E be the event "one of the balls drawn is 1", and F the event "the final sum of numbers on the balls drawn is even". Give the set of outcomes corresponding to each of the following events

      (i) (1 point) "both E and F occur"

      (ii) (1 point) "neither E nor F occurs"

Solution.

(a) The sample space S contains exactly the following outcomes: $(1,2,3)$, $(1,2,4)$, $(1,3,2)$, $(1,3,4)$, $(1,4)$, $(2,1,3)$, $(2,1,4)$, $(2,3)$, $(2,4)$, $(3,1,2)$, $(3,1,4)$, $(3,2)$, $(3,4)$, $(4,1)$, $(4,2)$, $(4,3)$.

(b) The event E, which is defined to be "one of the balls drawn is 1", when expressed as a subset of the sample space is

$$E = \{(1,2,3),(1,2,4),(1,3,2),(1,3,4),(1,4),(2,1,3),(2,1,4),(3,1,2),(3,1,4),(4,1)\}.$$

The event F, *i.e.*, "the final sum of numbers on the balls drawn is even", is the set

$$F = \{(1,2,3),(1,3,2),(1,3,4),(2,1,3),(2,4),(3,1,2),(3,1,4),(4,2)\}.$$

(i) The event "both E and F occur" is the set

$$E \cap F = \{(1,2,3),(1,3,2),(1,3,4),(2,1,3),(3,1,2),(3,1,4)\}.$$

(ii) The event "neither E nor F occurs" is the set

$$E^c \cap F^c = (E \cup F)^c = \{(2,3),(3,2),(3,4),(4,3)\}.$$

4. Rooks (or castles) are chess pieces that are only allowed to move horizontally or vertically on a chessboard. The following question on rook placements on a standard chessboard assumes that rooks must be placed on one of the 64 allowed positions on the board, and no two rooks can share the same position.

How many different ways are there of placing 8 indistinguishable rooks on a standard 8×8 chessboard

(a) (2 points) if there are no restrictions on their placement, other than those stated above?

(b) (2 points) if no rook can be in a position that attacks another?
[A rook can attack another if and only if they are on the same row or column of the chessboard.]

Solution.

(a) The number of unrestricted placements of 8 indistinguishable rooks on a standard chessboard is equal to the number of ways of picking 8 squares on the chessboard to place the rooks. This number is simply $\binom{64}{8}$.

(b) The restriction here is that no two rooks can be in the same row or column of the chessboard. Again, we do the count for the case of distinguishable rooks first, then divide by 8! to get the count for the indistinguishable case.

The number of ways of placing rook 1 is still 64, but once this rook has been placed, say at position (i,j), no further rooks can be placed on the ith row and jth column.

This means that there are only 49 ($= 7^2$) squares left on which rook 2 can possibly be placed. Eliminating the rows and columns containing rooks 1 and 2 leaves 36 ($= 6^2$) squares for rook 3, and so on until rook 8, which can only be placed on the one uncovered square remaining on the board. So, the number of non-attacking placements of 8 distinguishable rooks is $8^2 \times 7^2 \times 6^2 \times \cdots \times 1^2 = (8!)^2$. Dividing this by 8!, we find the number of non-attacking placements of 8 indistinguishable rooks to be 8!.

Here is another way of doing this count. Since there are 8 columns and 8 rooks to be placed, and there can be at most one rook in each column, we see that each column must in fact have exactly one rook. The number of positions a rook could go in the 1st column is 8; after the first column is filled, the number of positions a rook could go in the 2nd column is 7 (it cannot be placed in the same row as the rook in column 1); and so on, all the way down to the 8th column. Thus, the total number of placements of 8 non-attacking rooks is $8 \times 7 \times \cdots \times 1 = 8!$.

5. A child has 12 blocks, of which 6 are black, 4 are red, 1 is white, and 1 is yellow.

   (a) (2 points) If the child puts the blocks in a line, how many different arrangements are possible?

   (b) (2 points) If one of the arrangements in part (a) is randomly selected, what is the probability that no two black blocks are next to each other.

Solution.

(a) The number of ways the blocks can be arranged in a line is the number of distinguishable permutations of 12 objects of 4 different types such that 6 are type 1 (black), 4 are type 2 (red), 1 is type 3 (white), and 1 is type 4 (yellow). The number of all such distinguishable permutations is

$$\frac{12!}{6!4!1!1!} = 27720.$$

(b) Since there are 12 blocks, six of which are black, the condition that no two black blocks are neighbors means that between every two consecutive black blocks there is at least one block of a different color. The line either starts with a black block or with a block of different color. If the line starts with a different color, then between two black blocks there must be exactly one block of different color. If the line starts with a black block, then either black and different colors alternate (and the line ends with a different color), or there exist exactly two black blocks between which there are exactly two blocks of different color, and in the rest of the line the black and differently colored blocks alternate (the line ends with a black block). In the latter case, there are 5 ways of choosing where to insert the two blocks of different color between the two black blocks. Therefore, if we consider the non-black blocks indistinguishable, there are 7 different arrangements such that no two black blocks are next to each other. In all 7 cases, the 6 non-black blocks (now viewed as 4 red, 1 white, and 1 yellow) can be arranged in

$$\frac{6!}{4!1!1!} = 30$$

different ways. Thus the number of arrangements in question is $7 \cdot 30 = 210$ and the desired probability is

$$\frac{210}{27720} \approx 0.00757.$$

6. The old TV game *Let's Make a Deal* hosted by Monty Hall could be summarized as follows. Suppose you are on a game show, and you are given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say number 1, and the host, who knows what is behind the doors, opens another door, say number 3, which has a goat. He says to you, "Do you want to pick door number 2?" Is it to your advantage to switch your choice of doors?

   (a) (2 points) Assume that the host's protocol is: he is determined to show you a goat and with a choice of two, he picks one at random. Let p be the conditional probability that the third door conceals the car. Compute p.

   (b) (2 points) Assume that the host's protocol is: he is determined to show you a goat and with a choice of two goats (Dolly and Molly, say) he shows you Dolly with probability b. Compute p given you see Dolly.

   Solution.

   Let's assume that each of the 6 orderings of the goats and the car are equally likely. Let C_i be the event that the ith door conceals the car, C_w be the event that the third door (i.e., the door not chosen by you originally and not opened by the host) conceals the car, i.e., the event that the third door is a winning door, G the event that you see the goat, and D the event that you see Dolly. Suppose that you pick door 1 and the host opens another door. Using conditional probability we have

   (a)

$$\begin{aligned} p = P(C_w|G) &= \frac{P(C_wG)}{P(G)} = \frac{P(C_wG|C_1)P(C_1) + P(C_wG|C_1^c)P(C_1^c)}{P(G|C_1)P(C_1) + P(G|C_1^c)P(C_1^c)} \\ &= \frac{0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3}}{1 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3}} = \frac{2}{3} \end{aligned}$$

   or a simpler approach

$$p = P(C_w|G) = P(C_w) = P(C_w|C_1)P(C_1) + P(C_w|C_1^c)P(C_1^c) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}.$$

   Note that this part can also be solved without conditional probability. The contestant chooses a door. Consider the space of two events. E is the event the car is behind the door chosen by the contestant. E^c is the event the car is not behind the chosen door. $P(E) = 1/3 = P(E|\text{given Monty will open a door without a car}) = P(E|\text{given Monty has opened a door without a car})$. By not switching the contestant has a 1/3 chance of getting the car and by switching he has a 2/3 chance of getting the car.

(b) Use the similar formula and note $P(C_w D|C_1^c) = P(D|C_1^c) = 1/2$. We have

$$\begin{aligned} p = P(C_w|D) &= \frac{P(C_w D)}{P(D)} = \frac{P(C_w D|C_1)P(C_1) + P(C_w D|C_1^c)P(C_1^c)}{P(D|C_1)P(C_1) + P(D|C_1^c)P(C_1^c)} \\ &= \frac{0 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3}}{b \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3}} = \frac{1}{b+1}. \end{aligned}$$

7. (4 points) A box contains three coins, one of which is fair, one double-headed (*i.e.*, heads on both sides), and the third is biased in such a way that it comes up heads with probability 3/4. A coin is drawn at random from the box and flipped twice. If both flips result in heads, what is the probability that the coin drawn was double-headed?

Solution.

Let F, D and B be the events "fair coin selected", "double-headed coin selected" and "biased coin selected", respectively. Also, let HH be the event "both flips of the selected coin result in heads". We are asked to determine $P(D|HH)$.

From the information given to us, we have $P(F) = P(D) = P(B) = 1/3$, and (assuming that the two coin flips are independent)

$$P(HH|F) = (1/2)^2 = 1/4, \quad P(HH|D) = 1, \quad P(HH|B) = (3/4)^2 = 9/16.$$

So, applying Bayes' rule, we find

$$\begin{aligned} P(D|HH) &= \frac{P(HH|D)P(D)}{P(HH|F)P(F) + P(HH|D)P(D) + P(HH|B)P(B)} \\ &= \frac{1 \times 1/3}{(1/4 \times 1/3) + (1 \times 1/3) + (9/16 \times 1/3)} = \frac{16}{29}. \end{aligned}$$