

University of Ottawa
MAT 1330D First Midterm Exam

February 10, 2010. Duration: 80 minutes. Instructor: Catalin Rada

Family Name: _____

S O L

First Name: _____



Do **not** write your student ID number on this front page. Please write your student ID number in the space provided on the second page.

Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more points than others. Make a note of the questions that you feel confident you can do, and then do those first: you do not have to proceed through the paper in the order given.

- You have 80 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed. The use of cell phones, pagers or any text storage or communication device **is not permitted**.
- Only the Faculty approved TI-30 calculator is allowed.
- The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct. Answer these questions in the space provided. Use the backs of pages if necessary.
- Where it is possible to check your work, do so.
- Good Luck!

Student number: _____, Total marks: _____ out of 29

Problem	1	2	3	4	5	6
Marks						

Question 1. [4 points] Find the derivative of the following function:

$$f(x) = \frac{x^3 - x^{-3}}{x^4 + x^{-4}} + \ln(x)e^x.$$

Solution: By product rule, quotient rule and power rule one gets $f'(x) = \frac{(3x^2+3x^{-4})(x^4+x^{-4})-(x^3-x^{-3})(4x^3-4x^{-5})}{(x^4+x^{-4})^2} + \frac{1}{x}e^x + \ln(x)e^x.$

Question 2. [5 points] Over the course of a year, Ottawa receives a maximum of 14 hours of daylight and a minimum of 10. The maximum occurs in June, the minimum in December. Assume that the number of hours of daylight varies sinusoidally over a period of one year. Find the parameters in the standard cosine description, i.e.,

$$f(t) = A + B \cos(2\pi(t - \Phi)/T),$$

where t is in months, and $t = 0$ correspond to the month of January. Draw the graph of the function and identify the four parameters A, B, Φ, T in the graph. Give the names of the four parameters.

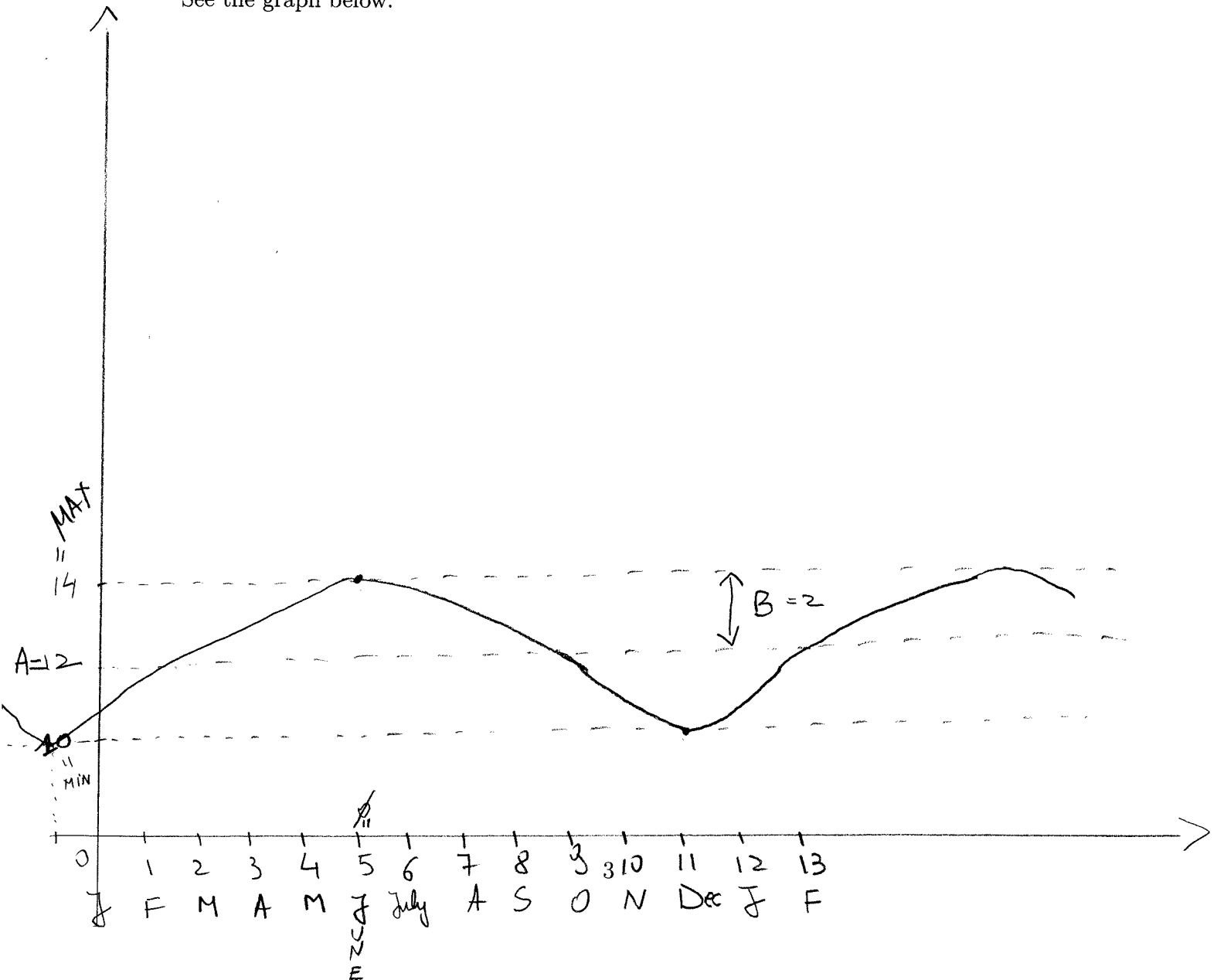
Solution: $A = \frac{14+10}{2} = \frac{24}{2} = 12$ average

$B = 14 - A = 14 - 12 = 2$ amplitude

$T = 12$ period

$\Phi = 5$ phase

See the graph below.



Question 3. [4 points] Consider the function $f(x) = \frac{3x}{(3-x)^4}$. Find the following limits if they exist. If not, indicate why.

(a) $\lim_{x \rightarrow 3^-} f(x)$, (b) $\lim_{x \rightarrow 3^+} f(x)$, (c) $\lim_{x \rightarrow 3} f(x)$, (d) $\lim_{x \rightarrow 2} f(x)$.

Solution: (a) $\lim_{x \rightarrow 3^-} \frac{3x}{(3-x)^4} = +\infty$ since $(3-x)^4 \rightarrow 0$ (as $x \rightarrow 3^-$) and $(3-x)^4 > 0$.

(b) $\lim_{x \rightarrow 3^+} \frac{3x}{(3-x)^4} = +\infty$ since $(3-x)^4 \rightarrow 0$ (as $x \rightarrow 3^+$) and $(3-x)^4 > 0$.

(c) $\lim_{x \rightarrow 3} \frac{3x}{(3-x)^4} = +\infty$ from (a) and (b).

(d) $\lim_{x \rightarrow 2} \frac{3x}{(3-x)^4} = \frac{3 \times 2}{(3-2)^4} = 6$.

Question 4. [4 points] Use **the definition of the derivative** to calculate the derivative of the function $f(x) = -(x - 1)^2 + 2010x$.

$$\begin{aligned} \text{Solution: } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{-(x+h-1)^2 + 2010(x+h) + (x-1)^2 - 2010x}{h} = \\ & \lim_{h \rightarrow 0} \frac{(x-1-x-h+1)(x-1+x+h-1) + 2010h}{h} = \lim_{h \rightarrow 0} \frac{h\{-2x+h-2\} + 2010}{h} = \\ & \lim_{h \rightarrow 0} \{-2x+h-2\} + 2010 = -(2x-2) + 2010 \text{ or } -2(x-1) + 2010. \end{aligned}$$

Question 5. [8 points] In order to keep the songbirds in the back yard happy, one person puts out 30g of seeds at the beginning of each week. During the week, the birds find and eat 1/4 of the available seeds. The DTDS for the amount of seeds in the back yard is

$$S_{t+1} = 0.25S_t + 30,$$

where t is measured in weeks and seeds are counted just before a new supply is provided.

- (a) [1 point] What is the updating function of the DTDS?
- (b) [1 point] Find the equilibrium point of the DTDS if there is one.
- (c) [2 points] Find the general solution formula for the DTDS, i.e., $S_t = \dots$
- (d) [2 points] Graph the updating function and draw the cobwebbing, starting from $S_0 = 5$, for at least 4 steps.
- (e) [2 points] Is the equilibrium point stable or not? Why?

Solution: (a) $f(S_t) = 0.25S_t + 30$

(b) The equation $S = 0.25S + 30$ gives us $0.75S = 30$, so $S = \frac{30}{0.75} = 40$

(c) From our second lecture one gets that $S_t = (0.25)^t S_0 + 30 \frac{(0.25)^t - 1}{0.25 - 1} =$

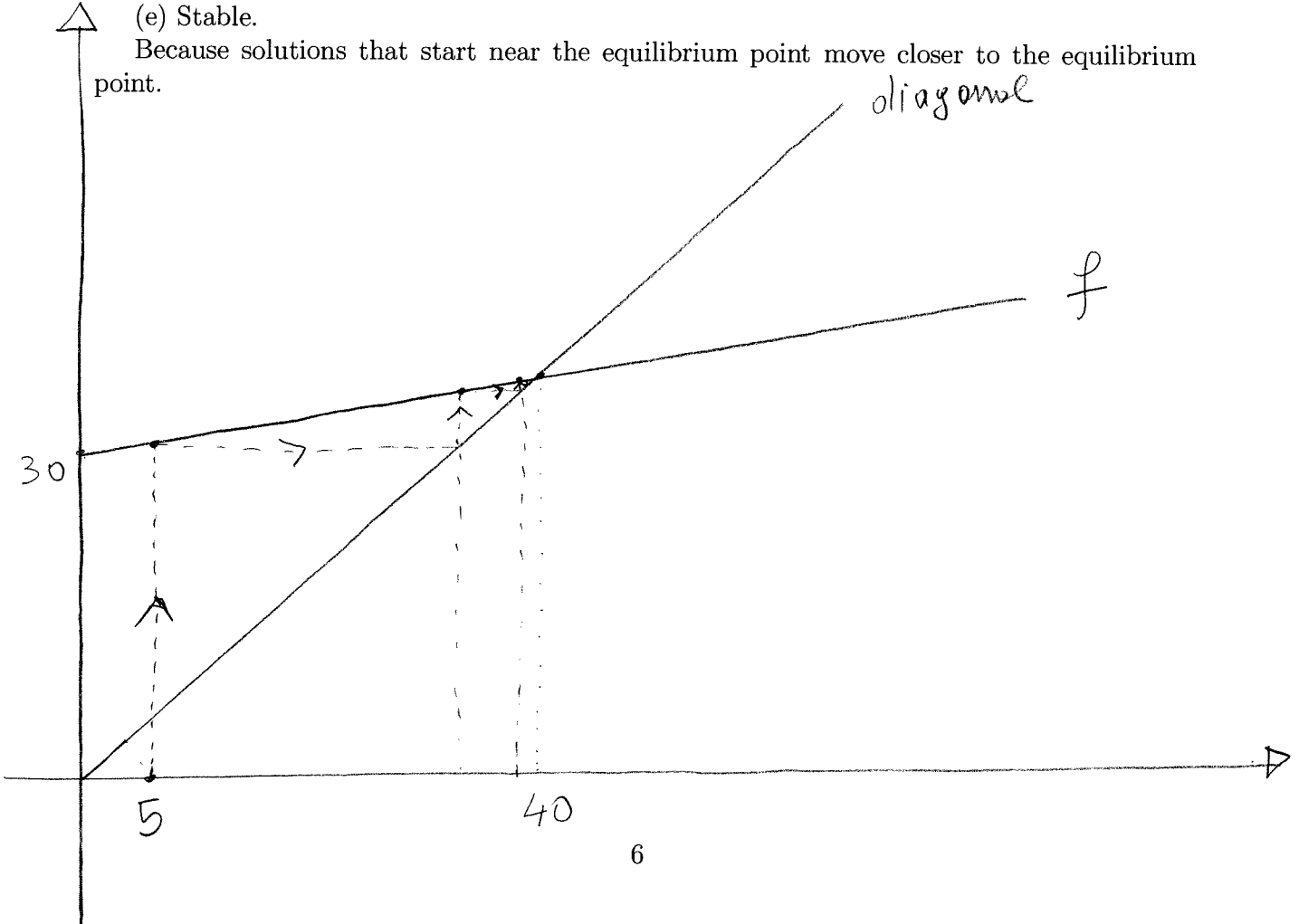
$$(0.25)^t S_0 + 30 \frac{-(0.25)^t + 1}{0.75} =$$

$$(0.25)^t S_0 + 40(1 - (0.25)^t) = (0.25)^t (S_0 - 40) + 40 \text{ where } S_0 \text{ is the initial condition.}$$

(d) See below.

(e) Stable.

Because solutions that start near the equilibrium point move closer to the equilibrium point.



Question 6. [4 points] a) Consider the function f given below:

$$f(x) = \begin{cases} 3x^2 + a + 2010 & \text{for } x \geq 10, \\ \ln(e^{x^2} + x - 10) & \text{for } 0 \leq x < 10. \end{cases}$$

Find the values of a for which f is a continuous function.

b) Consider the function g given below:

$$g(x) = -\frac{x^3}{3} - \frac{x^2}{2} + 2x + 2010.$$

Find the critical points of g . Determine the intervals where g' is positive and the intervals where g' is negative.

Solution: a) f is continuous if and only if $\lim_{x \rightarrow 10^-} f(x) = \lim_{x \rightarrow 10^+} f(x) = f(10)$. Note that $\lim_{x \rightarrow 10^-} f(x) = \lim_{x \rightarrow 10^-} \ln(e^{x^2} + x - 10) = \ln(e^{10^2} + 10 - 10) = \ln(e^{10^2}) = 10^2 = 100$.

Note that $\lim_{x \rightarrow 10^+} f(x) = \lim_{x \rightarrow 10^+} 3x^2 + a + 2010 = 3 \times 10^2 + a + 2010$.

From $100 = 300 + a + 2010$ one gets that $a = -2210$.

b) Note that by power rule one gets that $g'(x) = -x^2 - x + 2 = -(x^2 + x - 2) = -(x + 2)(x - 1)$. So g is everywhere differentiable!

The critical points are obtained from the equation $-(x + 2)(x - 1) = 0$, so they are: $x_1 = -2$ and $x_2 = 1$.

Since we are dealing with a quadratic one gets that:

g' is positive if x is in the interval $(-2, 1)$;

g' is negative if x is either in the interval $(-\infty, -2)$ or in the interval $(1, \infty)$.

Student number: _____, Total marks: _____ out of 29

Problem	1	2	3	4	5	6
Marks						

Question 1. [4 points] Find the derivative of the following function:

$$f(x) = \frac{x^4 - x^{-4}}{x^3 + x^{-3}} + e^x \ln(x).$$

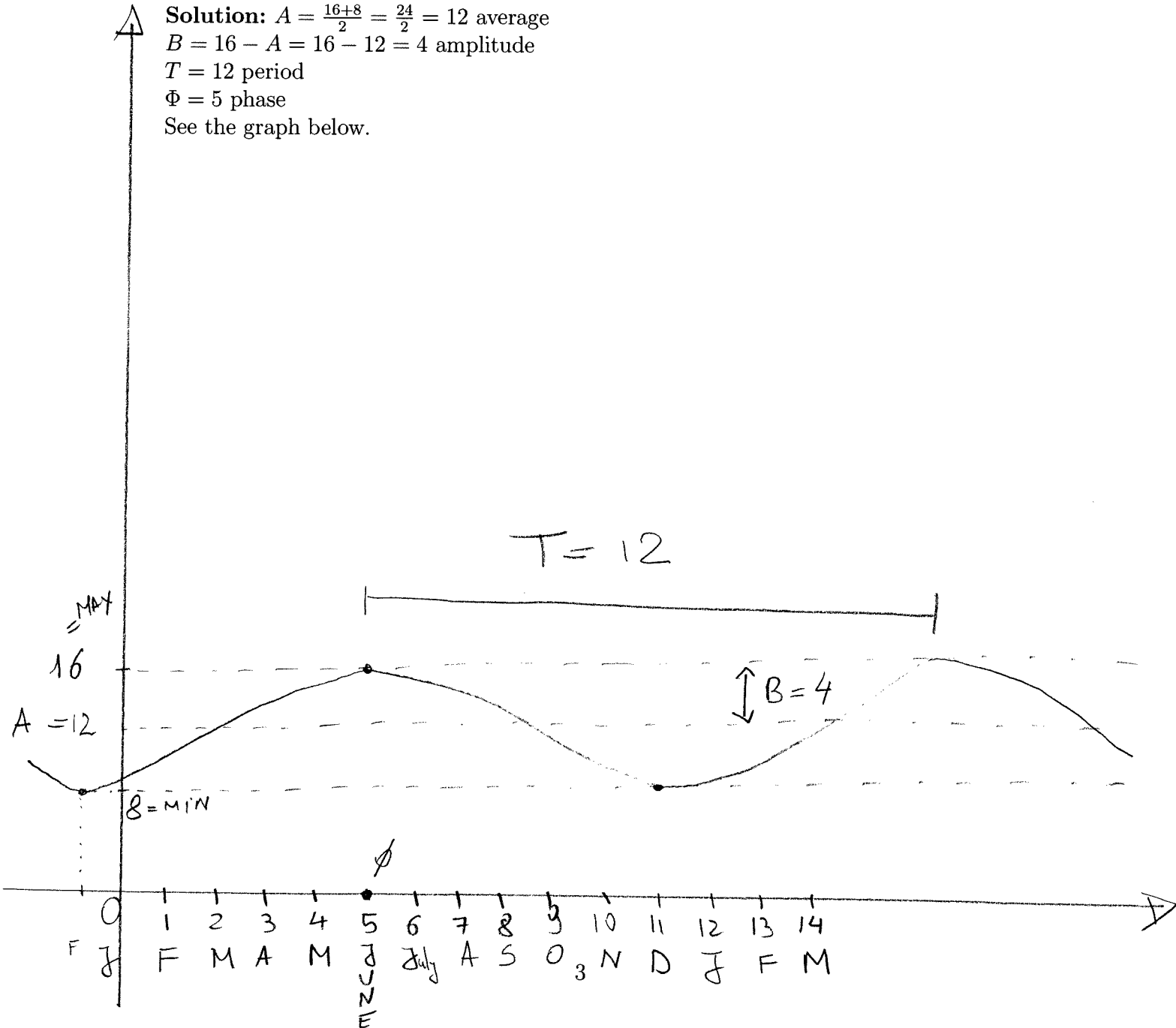
Solution: By product rule, quotient rule and power rule one gets $f'(x) = \frac{(4x^3 + 4x^{-5})(x^3 + x^{-3}) - (x^4 - x^{-4})(3x^2 - 3x^{-4})}{(x^3 + x^{-3})^2} + e^x \ln(x) + \frac{1}{x}e^x$.

Question 2. [5 points] Over the course of a year, Ottawa receives a maximum of 16 hours of daylight and a minimum of 8. The maximum occurs in June, the minimum in December. Assume that the number of hours of daylight varies sinusoidally over a period of one year. Find the parameters in the standard cosine description, i.e.,

$$f(t) = A + B \cos(2\pi(t - \Phi)/T),$$

where t is in months, and $t = 0$ correspond to the month of January. Draw the graph of the function and identify the four parameters A, B, Φ, T in the graph. Give the names of the four parameters.

Solution: $A = \frac{16+8}{2} = \frac{24}{2} = 12$ average
 $B = 16 - A = 16 - 12 = 4$ amplitude
 $T = 12$ period
 $\Phi = 5$ phase
 See the graph below.



Question 3. [4 points] Consider the function $f(x) = \frac{3x}{(3-x)^5}$. Find the following limits if they exist. If not, indicate why.

$$(a) \lim_{x \rightarrow 3^-} f(x), \quad (b) \lim_{x \rightarrow 3^+} f(x), \quad (c) \lim_{x \rightarrow 3} f(x), \quad (d) \lim_{x \rightarrow 2} f(x).$$

Solution: (a) $\lim_{x \rightarrow 3^-} \frac{3x}{(3-x)^5} = +\infty$ since $(3-x)^5 \rightarrow 0$ (as $x \rightarrow 3^-$) and $(3-x)^5 > 0$.

(b) $\lim_{x \rightarrow 3^+} \frac{3x}{(3-x)^5} = -\infty$ since $(3-x)^5 \rightarrow 0$ (as $x \rightarrow 3^+$) and $(3-x)^5 < 0$.

(c) $\lim_{x \rightarrow 3} \frac{3x}{(3-x)^5}$ does not exist (see (a) and (b)).

(d) $\lim_{x \rightarrow 2} \frac{3x}{(3-x)^5} = \frac{3 \times 2}{(3-2)^5} = 6$.

Question 4. [4 points] Use **the definition of the derivative** to calculate the derivative of the function $f(x) = -(x - 1)^2 + 2010$.

$$\begin{aligned} \text{Solution: } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{-(x+h-1)^2 + 2010 + (x-1)^2 - 2010}{h} = \\ & \lim_{h \rightarrow 0} \frac{(x-1-x-h+1)(x-1+x+h-1)}{h} = \lim_{h \rightarrow 0} \frac{h\{-(2x+h-2)\}}{h} = \\ & \lim_{h \rightarrow 0} \{-(2x+h-2)\} = -(2x-2) \text{ or } -2(x-1). \end{aligned}$$

Question 5. [8 points] In order to keep the songbirds in the back yard happy, one person puts out 40g of seeds at the beginning of each week. During the week, the birds find and eat 1/5 of the available seeds. The DTDS for the amount of seeds in the back yard is

$$S_{t+1} = 0.2S_t + 40,$$

where t is measured in weeks and seeds are counted just before a new supply is provided.

- [1 point] What is the updating function of the DTDS?
- [1 point] Find the equilibrium point of the DTDS if there is one.
- [2 points] Find the general solution formula for the DTDS, i.e., $S_t = \dots$
- [2 points] Graph the updating function and draw the cobwebbing, starting from $S_0 = 5$, for at least 4 steps.
- [2 points] Is the equilibrium point stable or not? Why?

Solution: (a) $f(S_t) = 0.2S_t + 40$

(b) The equation $S = 0.2S + 40$ gives us $0.8S = 40$, so $S = \frac{40}{0.8} = 50$

(c) From our second lecture one gets that $S_t = (0.2)^t S_0 + 40 \frac{(0.2)^t - 1}{0.2 - 1} =$

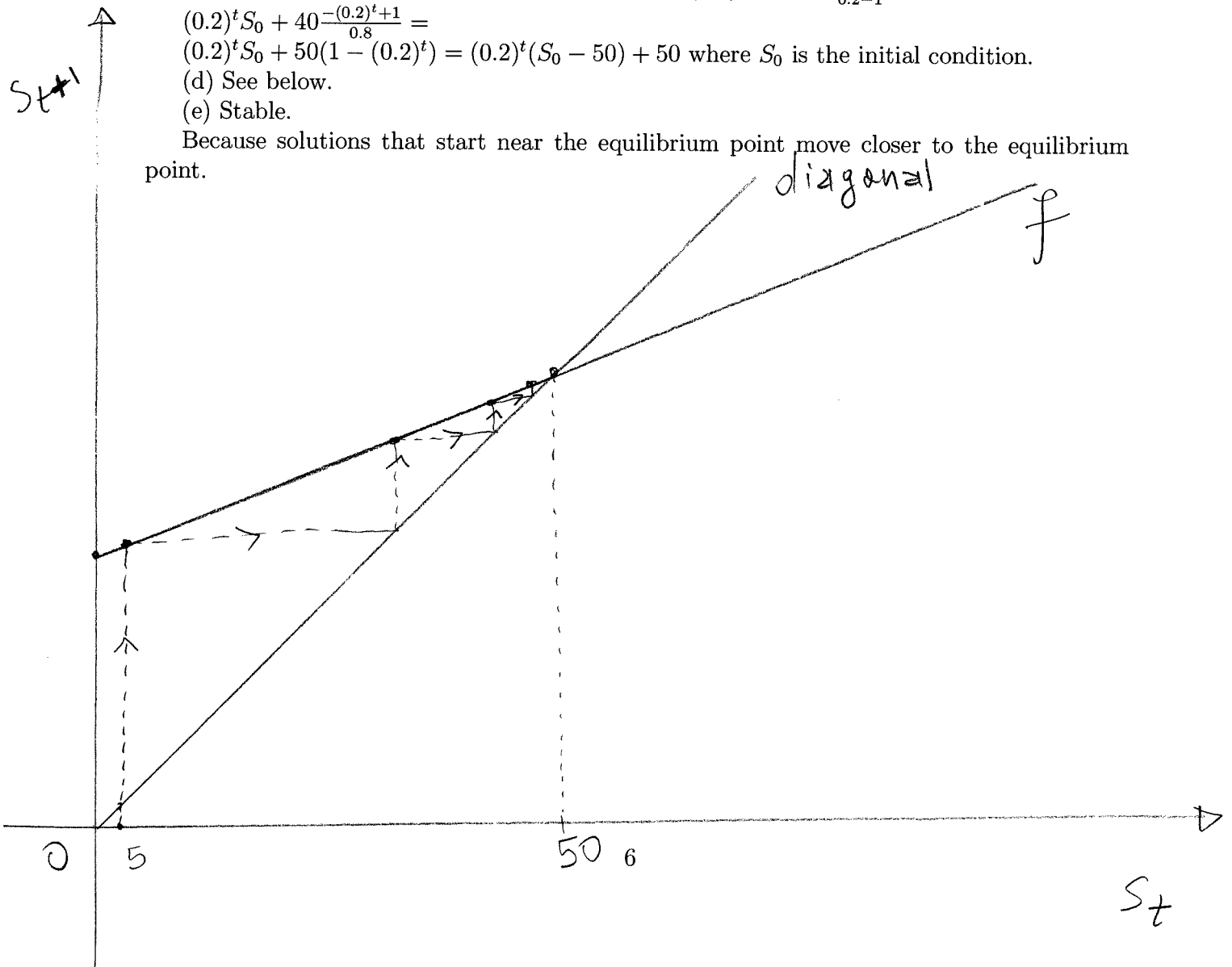
$$(0.2)^t S_0 + 40 \frac{-(0.2)^t + 1}{0.8} =$$

$$(0.2)^t S_0 + 50(1 - (0.2)^t) = (0.2)^t (S_0 - 50) + 50 \text{ where } S_0 \text{ is the initial condition.}$$

(d) See below.

(e) Stable.

Because solutions that start near the equilibrium point move closer to the equilibrium point.



Question 6. [4 points] a) Consider the function f given below:

$$f(x) = \begin{cases} 3x^2 + a + 2010 & \text{for } x \geq 11, \\ \ln(e^{x^2} + x - 11) & \text{for } 0 \leq x < 11. \end{cases}$$

Find the values of a for which f is a continuous function.

b) Consider the function g given below:

$$g(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x - 2010.$$

Find the critical points of g . Determine the intervals where g' is positive and the intervals where g' is negative.

Solution: a) f is continuous if and only if $\lim_{x \rightarrow 11^-} f(x) = \lim_{x \rightarrow 11^+} f(x) = f(11)$. Note that $\lim_{x \rightarrow 11^-} f(x) = \lim_{x \rightarrow 11^-} \ln(e^{x^2} + x - 11) = \ln(e^{11^2} + 11 - 11) = \ln(e^{11^2}) = 11^2 = 121$.

Note that $\lim_{x \rightarrow 11^+} f(x) = \lim_{x \rightarrow 11^+} 3x^2 + a + 2010 = 3 \times 11^2 + a + 2010$.

From $121 = 363 + a + 2010$ one gets that $a = -2252$.

b) Note that by power rule one gets that $g'(x) = x^2 + x - 2 = (x + 2)(x - 1)$. So g is everywhere differentiable!

The critical points are obtained from the equation $(x + 2)(x - 1) = 0$, so they are: $x_1 = -2$ and $x_2 = 1$.

Since we are dealing with a quadratic one gets that:

g' is negative if x is in the interval $(-2, 1)$;

g' is positive if x is either in the interval $(-\infty, -2)$ or in the interval $(1, \infty)$.